Linear Differential Equations with the Hukuhara Derivative that Preserve Sets of Constant Width

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Solutions of ordinary differential equations with the Hukuhara derivative [1], [2, p. 14] are compact convex sets for every value of the independent variable. The geometric characteristics of these sets can be considered as functions of the independent variable. The study of these functions is of particular interest. For example, in the paper [4] the radii of inscribed and circumscribed spheres of solutions of linear time-invariant differential equations were considered and their Lyapunov exponents were calculated. The paper [3] gives a complete description of linear time-invariant differential equations with the Hukuhara derivative that preserve polytopes, i.e., of equations such that any solution of them that is a polytope for the initial value of the independent variable remains a polytope for all subsequent values.

This report examines other geometric characteristics of solutions. But before formulating the obtained result, let us give some necessary definitions. By $\Omega(\mathbb{R}^d)$ we denote the family of all nonempty bounded subsets of the space $\mathbb{R}^d$. The set of all nonempty convex compact subsets of the space $\mathbb{R}^d$ is denoted by $K_c(\mathbb{R}^d)$.

Definition 1. A set $X \subset K_c(\mathbb{R}^d)$ is called a *set of constant width* if the length of the orthogonal projection of X onto an arbitrary line equals the same value $w(X)$ that is called the *width* of X.

Definition 2. A set $Z \overset{\text{def}}{=} \{x + y : \ x \in X, \ y \in Y\}$ is called the *Minkowski sum* of two subsets X, $Y \subset \mathbb{R}^d$.

Generally speaking, for arbitrary real matrices A and B consisting of d columns and a set $X \subset \mathbb{R}^d$, we have $(A + B)X \neq AX + BX$.

Definition 3 ([1])**.** A set $Z \subset \mathbb{R}^d$ is called the *Hukuhara difference* of X, $Y \subset \mathbb{R}^d$ and denoted by $Z = X - Y$, if $X = Y + Z$.

By $B \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \ \|x\| \leq 1\}$ we denote the closed ball of unit radius centered at the origin.

Definition 4. The *Hausdorff distance* $h(\,\cdot\,,\,\cdot\,)$ on the set $\Omega(\mathbb{R}^d)$ is the function

$$h(X, Y) \overset{\text{def}}{=} \inf\big\{r \geq 0 : \ X \subset Y + rB, \ Y \subset X + rB\big\}, \quad X, Y \in \Omega(\mathbb{R}^d).$$

According to Hahn Theorem, the pair $\big(K_c(\mathbb{R}^d), h\big)$ is a complete metric space. By $I \subset \mathbb{R}$ we denote an arbitrary open interval that may be unbounded.

Definition 5 ([1])**.** A mapping $X : I \to K_c(\mathbb{R}^d)$ is called *differentiable by Hukuhara* at a point $t_0 \in I$ if there exist limits

$$\lim_{\Delta t \to +0} \frac{X(t_0 + \Delta t) - X(t_0)}{\Delta t}, \quad \lim_{\Delta t \to +0} \frac{X(t_0) - X(t_0 - \Delta t)}{\Delta t}$$

and these limits are equal to each other. In this case, the common value of these limits, which is obviously a convex compact set, is denoted by $D_H X(t_0)$ and called the Hukuhara derivative of the mapping X at the point t_0.

Consider the linear differential equation

$$D_H X = \sum_{i=1}^{n} A_i(t)X, \quad X(t) \in K_c(\mathbb{R}^d), \ \ t \geq 0, \tag{1}$$

with piecewise continuous $d \times d$-matrices of coefficients $A_i(\,\cdot\,)$, $1 \leq i \leq n$. We say that equation (1) *preserves sets of constant width* if for any its solution $X(\,\cdot\,)$, such that $X(0)$ is a set of constant width, it follows that $X(t)$ is a set of constant width for all $t \geq 0$. Naturally, the problem arises of obtaining a necessary and sufficient condition for equation (1) to preserve sets of constant width. The complete solution of the problem is given by the following theorem.

Theorem. *Equation* (1) *preserves sets of constant width if and only if there exist piecewise continuous function* $\alpha_1(\,\cdot\,), \alpha_2(\,\cdot\,), \dots, \alpha_n(\,\cdot\,) : [0, \infty) \to \mathbb{R}_{\geq 0}$*, such that the following equalities hold*

$$A_i(t)^T A_i(t) = \alpha_i(t)E, \ \ 1 \leq i \leq n, \ \ t \geq 0.$$

References

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