On the Existence of an Optimal Element in Control Problems with Several Constant Delays

Tamaz Tadumadze[1,2]

[1] *I. Javakhishvili Tbilisi State University, Department of Mathematics, Tbilisi, Georgia*
[2] *I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University Tbilisi, Georgia*
E-mail: tamaz.tadumadze@tsu.ge

In the paper, Filippov's type theorems on the existence of an optimal element [2] are given for the nonlinear optimal control problems with delays in the phase coordinates and commensurable delays in controls. Unlike considered in [1,3,5–9], here under element is implied the collection of the final moment t_1, the delay parameters τ_i, $i=1,\dots,s$ containing in the phase coordinates, the initial vector x_0, the piecewise-continuous initial function $\varphi(t)$ and measurable control function $u(t)$.

Let $I=[t_0,T]$ be a fixed interval; let $\tau_{2i}>\tau_{1i}>0$, $i=1,\dots,s$ and $\theta_p>\cdots>\theta_1>0$ be given numbers. Suppose that $O\subset\mathbb{R}^n$ is an open set and $\Phi\subset O$ and $U\subset\mathbb{R}^r$ are compact sets; the function $f(t,x_0,x_1,\dots,x_s,u_0,u_1,\dots,u_p)$ is continuous on the set $I\times O^{1+s}\times U^{1+p}$ and continuously differentiable with respect to $x_i\in O$, $i=0,1,\dots,s$; denote by Δ the set of piecewise-continuous functions $\varphi:[t_0-\tau,t_0]\to\Phi$, where $\tau=\max\{\tau_{21},\dots,\tau_{2s}\}$, satisfying the conditions:

(a) for each function $\varphi(t)\in\Delta$ there exists a partition $t_0-\tau=\xi_0<\xi_1<\cdots<\xi_{k+1}=t_0$ of the interval $[t_0-\tau,t_0]$ such that the restriction of the function $\varphi(t)$ satisfies Lipschitz's condition on the open interval (ξ_i,ξ_{i+1}), $i=0,1,\dots,k$, i.e.

$$|\varphi(s_1)-\varphi(s_2)|\le L|s_1-s_2|,\ \ \forall\, s_1,s_2\in(\xi_i,\xi_{i+1}),\ \ i=0,1,\dots,k;$$

(b) the numbers k and L do not depend on $\varphi(t)$. By Ω we denote the set of measurable functions $u:[t_0-\theta_p,T]\to U$. Let

$$g_i: I\times[\tau_{11},\tau_{21}]\times\cdots\times[\tau_{1s},\tau_{2s}]\times X_0\times O\to\mathbb{R}^1,\ \ i=0,1,\dots,l$$

be continuous functions, where $X_0\subset O$ is a compact set. In the space $\mathbb{R}^n$ to each element

$$w=(t_1,\tau_1,\dots,\tau_s,x_0,\varphi(t),u(t))\in W=(t_0,T]\times[\tau_{11},\tau_{21}]\times\cdots\times[\tau_{1s},\tau_{2s}]\times X_0\times\Delta\times\Omega$$

we assign the differential equation with delays in the phase coordinates and controls

$$\dot x(t)=f\big(t,x(t),x(t-\tau_1),\dots,x(t-\tau_s),u(t),u(t-\theta_1),\dots,u(t-\theta_p)\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[t_0-\tau,t_0),\quad x(t_0)=x_0. \tag{2}$$

Definition 1. Let $w=(t_1,\tau_1,\dots,\tau_s,x_0,\varphi(t),u(t))\in W$. A function $x(t)=x(t;w)\in O$, $t\in[t_0-\tau,t_1]$, is called a solution corresponding to the element w, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies Eq. (1) almost everywhere on $[t_0,t_1]$.

Definition 2. An element $w \in W$ is said to be admissible if there exists the corresponding solution $x(t) = x(t; w)$, satisfying the condition

$$g(t_1, \tau_1, \dots, \tau_s, x_0, x(t_1)) = 0, \quad \text{where } g = (g_1, \dots, g_l). \tag{3}$$

By W_0 we denote the set of admissible elements. Now we consider the functional

$$J(w) = g_0(t_1, \tau_1, \dots, \tau_s, x_0, x(t_1; w)).$$

Definition 3. An element $w_0 = (t_{10}, \tau_{10}, \dots, \tau_{s0}, x_{00}, \varphi_0(t), u_0(t)) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{4}$$

(1)–(4) is called the control problem with delays in the phase coordinates and controls.

Theorem 1. *There exists an optimal element $w_0 \in W_0$ if the following conditions holds:*

1) *there exist a number $h > 0$ such that $\theta_i = ih$, $i = 1, \dots, p$ – commensurability of delays θ_i, $i = 1, \dots, p$;*

2) *$T = t_0 + mh$, where m is a natural number with $m \geq p$;*

3) *$W_0 \neq \varnothing$;*

4) *there exists a number $M > 0$ such that for an arbitrary $w \in W_0$,*

$$|x(t; w)| \leq M, \quad t \in [t_0, t_1];$$

5) *for each fixed $t \in [t_0, t_0 + h]$ and $z_i = (x_{0i}, x_{1i}, \dots, x_{si}) \in O^{1+s}$, $i = 0, 1, \dots, m-1$ the set*

$$V_f(t; z_0, z_1, \dots, z_{m-1}) := \left\{ \begin{pmatrix} f(t, z_0, u_0, u_{-1}, \dots, u_{-p}) \\ f(t+h, z_1, u_1, u_0, u_{-1}, \dots, u_{-p+1}) \\ \vdots \\ f(t+ph, z_p, u_p, u_{p-1}, \dots, u_0) \\ f(t+(p+1)h, z_{p+1}, u_{p+1}, u_p, \dots, u_1) \\ \vdots \\ f(t+(m-2)h, z_{m-2}, u_{m-2}, u_{m-3}, \dots, u_{m-p-2}) \\ f(t+(m-1)h, z_{m-1}, u_{m-1}, u_{m-2}, \dots, u_{m-p-1}) \end{pmatrix} : \ u_i \in U, \ i = -p, \dots, -1, 0, 1, \dots, m-1 \right\}$$

is convex.

Remark. Let U be a convex set and

$$f(t, x_0, x_1, \dots, x_s, u_0, u_1, \dots, u_p) = \sum_{i=0}^{p} A_i(t, x_0, x_1, \dots, x_s) u_i.$$

Then the condition 5) of Theorem 1 holds.

Now we consider an optimal problem with the integral functional and with fixed ends

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_p)\big), \ \ t \in I, \tag{5}$$

$$x(t) = \varphi(t), \ \ t \in [t_0-\tau, t_0), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{6}$$

$$\int\limits_{t_0}^{t_1} f_0\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t), u(t-\theta_1), \dots, u(t-\theta_p)\big)\, dt \longrightarrow \min. \tag{7}$$

Here $f_0(t, x_0, x_1, \dots, x_s, u_0, u_1, \dots, u_p) : I \times O^{1+s} \times U^{1+p} \to \mathbb{R}^1$ is a continuous function, $x_0, x_1 \in O$ are fixed points. For the problem (5)–(7) by Q_0 we denote the set of admissible elements

$$q = (t_1, \tau_1, \dots, \tau_s, \varphi(t), u(t)) \in Q = (t_0, T] \times [\tau_{11}, \tau_{21}] \times \dots \times [\tau_{1s}, \tau_{2s}] \times \Delta \times \Omega$$

and by

$$q_0 = (t_{10}, \tau_{10}, \dots, \tau_{s0}, \varphi_0(t), u_0(t))$$

we denote an optimal element(see Definitions 2 and 3).

Theorem 2. *There exists an optimal element $q_0 \in Q_0$ if the conditions 1) and 2) of Theorem 1 hold. Moreover: $Q_0 \neq \oslash$; there exists a number $M > 0$ such that for an arbitrary $q \in Q_0$ we have $|x(t;q)| \le M$, $t \in [t_0, t_1]$; for each fixed $t \in [t_0, t_0+h]$ and $z_i = (x_{0i}, \dots, x_{si}) \in O^{1+s}, i = 0, 1, \dots, m-1$ the set $V_F(t; z_0, z_1, \dots, z_{m-1})$ is convex, where $F = (f_0, f)$.*

It is clear that Theorem 2 is valid also for a problem with the free right end. Below we give an example which shows that for the existence of an optimal element the convexity of the set V_F is essential.

Example. Consider the optimal control problem

$$\dot{x}(t) = -x\big(t-\sqrt{2}\big) + u(t) + u^2(t-1), \ \ t \in [0,2],$$

$$x(t) = 0, \ \ t \in \big[-\sqrt{2}, 0\big), \quad x(0) = 0; \quad u(t) = 1, \ \ t \in [-1, 0), \quad u(t) \in U = [-1, 1], \ \ t \in [0, 2],$$

$$\int\limits_0^2 [x(t) - t]^2\, dt \to \min.$$

Here under an element is implied only control function $u(t) \in \Omega$. For a given $i = 2, 3, \dots$ we shall decompose the interval $[0, 1]$ into intervals I_j, $j = 2, \dots, i$, of length $1/i$. Define the control $u_i(t)$, $t \in [0, 2]$:

$$\begin{aligned} &u_i(t) = v_i(t), \ \ t \in [0,1];\\ &u_i(t) = 0, \ \ t \in \big(1, \sqrt{2}\big];\\ &u_i(t) = t - \sqrt{2}\,, \ \ t \in \big(\sqrt{2}, 2\big], \end{aligned}$$

here $v_i(t)$ is a control oscillating between $+1$ and -1, i.e.

$$v_i(t) = 1, \ \ t \in I_1, \quad v_i(t) = -1, \ \ t \in I_2,$$

etc. Furthermore,

$$\lim_{i \to \infty} x(t; u_i) = x_0(t) = t \ \text{ uniformly in } \ [0, 2]$$

and by Gamkrelidze's approximation lemma [4] the sequence of Dirac measures $\delta_{v_i(t)}$, $i=2,3,\dots$, $t\in[0,1]$ weakly converges to $\frac{1}{2}\,\delta_{-1}+\frac{1}{2}\,\delta_{+1}$. It is easy to observe that the trajectory $x_0(t)$, $t\in[0,2]$, minimizing the functional corresponds to the control

$$u_0(t)=\begin{cases}0, & t\in[0,1],\\ 1, & t\in\big(1,\sqrt{2}\,\big],\\ t+1-\sqrt{2}\,, & t\in\big(\sqrt{2},2\big].\end{cases}$$

But $u_0(t)\notin[-1,1]$, $t\in(\sqrt{2},2]$, i.e. it is not an admissible control. Consequently, in the considered example, there is no optimal element since the set

$$V_F(t;z_0,z_1):=\left\{\begin{pmatrix}F(t,z_0,u_0,u_{-1})\\ F(t+1,z_1,u_1,u_0)\end{pmatrix}:\ \ u_i\in[-1,1],\ i=-1,0,1\right\}$$

is not convex. Here $z_i=(x_{0i},x_{1i})$, $i=0,1$ and

$$F(t,z_0,u_0,u_{-1})=\begin{pmatrix}(x_{00}-t)^2\\ -x_{10}+u_0+u_{-1}^2\end{pmatrix},\quad F(t+1,z_1,u_1,u_0)=\begin{pmatrix}(x_{01}-t-1)^2\\ -x_{11}+u_1+u_0^2\end{pmatrix}.$$

References

[1] T. S. Angell, Existence theorems for optimal control problems involving functional differential equations. *J. Optim. Theory Appl.* **7** (1971), 149–169.

[2] A. F. Filippov, On certain questions in the theory of optimal control. *J. SIAM Control Ser. A* **1** (1962), 76–84.

[3] R. Gabasov and F. Kirillova, *The Qualitative Theory of Optimal Processes.* (Russian) Monographs in Theoretical Foundations of Technical Cybernetics. Izdat. "Nauka", Moscow, 1971.

[4] R. V. Gamkrelidze, *Principles of Optimal Control Theory.* Mathematical Concepts and Methods in Science and Engineering, Vol. 7. Plenum Press, New York–London, 1978.

[5] G. L. Kharatishvili and T. A. Tadumadze, Formulas for the variation of a solution and optimal control problems for differential equations with retarded arguments. (Russian) *Sovrem. Mat. Prilozh.* No. 25, Optimal. Upr. (2005), 3–166; translation in *J. Math. Sci. (N.Y.)* **140** (2007), no. 1, 1–175

[6] T. A. Tadumadze, The existence of a solution in optimal problems with deviating argument. (Russian) *Sakharth. SSR Mecn. Akad. Moambe* **89** (1978), no. 2, 313–316.

[7] T. Tadumadze, Existence theorems for solutions of optimal problems with variable delays. *Control Cybernet.* **10** (1981), no. 3-4, 125–134 (1982).

[8] T. A. Tadumadze, *Some Problems in the Qualitative Theory of Optimal Control.* (Russian) Tbilis. Gos. Univ., Tbilisi, 1983.

[9] T. Tadumadze, On the existence of an optimal element for a delay control problem. *Nonlinear Anal.* **73** (2010), no. 1, 211–220.