Asymptotic Behaviour of Special Types of Solutions of Third Order Differential Equations, Asymptotically Close to Linear

A. A. Stekhun

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: angela.stehun@gmail.com

Consider the third order differential equation

$$y''' = \alpha_0 p(t) y L(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $L : \Delta_{Y_0} \to]0, +\infty[$ is a continuous function slowly varying as $y \to Y_0$, Y_0 is equal to either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0.

In the case where $L(y) \equiv 1$, Eq. (1) is a linear third-order differential equation. The asymptotic behavior of its solutions as $t \to +\infty$ (the case $\omega = +\infty$) is investigated in details (see, for example, the monograph [6, § 6, p . 175–194]).

Eq. (1) is a special case of the n-th order equation with regularly varying nonlinearity which was studied in work [2] (see also [3, 4]). However, the results of this work did not cover the case of an equation that is asymptotically close to linear. Some results on the asymptotic behavior of solutions of equation (1) were obtained in [5].

A second-order differential equation with a similar right-hand side was studied in paper [1].

The purpose of this work is to establish necessary and sufficient conditions for existence, as well as asymptotic representations of $P_\omega(Y_0, \lambda_0)$-solutions of the differential equation (1) in special cases, when $\lambda_0 \in \{0, 1, \pm\infty\}$.

Definition 1. The solution y of Eq. (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the conditions

$$y : [t_0, \omega[\to \Delta_{Y_0}, \quad \lim_{t\uparrow\omega} y(t) = Y_0,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty \end{cases} \quad (k = 1, 2), \quad \lim_{t\uparrow\omega} \frac{[y''(t)]^2}{y'''(t) y'(t)} = \lambda_0.$$

According to the properties of slowly varying functions (see [7]), for any function $L : \Delta_{Y_0} \to]0, +\infty[$, slowly varying as $y \to Y_0$, there exists a continuously differentiable, slowly varying as $y \to Y_0$ function $L_0 : \Delta_{Y_0} \to]0, +\infty[$ such that

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{L(y)}{L_0(y)} = 1 \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y L_0'(y)}{L_0(y)} = 0. \tag{2}$$

We set

$$\Delta_{Y_0} = \Delta_{Y_0}(b) = \begin{cases} [b, Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ ]Y_0, b], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0, \end{cases}$$

where a number $b \in \Delta_{Y_0}$ is such that

$$|b| < 1 \ \text{ as } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ as } \ Y_0 = +\infty \ \ (Y_0 = -\infty).$$

We introduce the following notation

$$\mu_0 = \operatorname{sign} b, \quad \mu_1 = \begin{cases} \mu_0, & \text{if } Y_0 = \pm\infty, \\ -\mu_0, & \text{if } Y_0 = 0, \end{cases}$$

that define respectively the signs of the $P_\omega(Y_0, \lambda_0)$-solution and its first derivative in some left neighborhood of ω. We also need the following functions

$$\Phi(y) = \int\limits_B^y \frac{ds}{sL^{\frac{1}{3}}(s)}\,, \quad B = \begin{cases} b, & \text{if } \displaystyle\int\limits_b^{Y_0} \frac{ds}{sL^{\frac{1}{3}}(s)} = \pm\infty, \\ Y_0, & \text{if } \displaystyle\int\limits_b^{Y_0} \frac{ds}{sL^{\frac{1}{3}}(s)} = const, \end{cases}$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \quad I_1(t) = \int\limits_{A_1}^t p(\tau)\,d\tau, \quad I_2(t) = \int\limits_{A_2}^t p^{\frac{1}{3}}(\tau)\,d\tau,$$

where each of the integration limits $A_i \in \{\omega; a\}$ $(i = 1, 2)$ is chosen so that the corresponding integral tends either to zero or $\pm\infty$ as $t \uparrow \omega$.

The function Φ is strictly monotone and differentiable on Δ_{Y_0}. For it there is a continuously differentiable and strictly monotone inverse function $\Phi^{-1} : \Delta_Z(c) \to \Delta_{Y_0}$, for which

$$\lim_{z\to Z} \Phi^{-1}(z) = Y_0, \quad Z = \lim_{y\to Y_0} \Phi(y),$$

where

$$\Delta_Z = \begin{cases} [c, Z[\,, & \text{if } \mu_0 > 0, \\ ]Z, c], & \text{if } \mu_0 < 0, \end{cases} \quad c = \Phi(b).$$

Theorem 1. *Let the function $L(\Phi^{-1}(z))$ be a regularly varying as $z \to Z$ of index γ. Then for the existence of $P_\omega(Y_0, 1)$-solutions of equation* (1) *it is necessary and, if function $p : [a, \omega[\to]0, +\infty[$ is continuously differentiable and there is the finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega} \frac{\big(p^{\frac{1}{3}}(t)L_0^{\frac{1}{3}}(\Phi^{-1}(\alpha_0 I_2(t)))\big)'}{p^{\frac{2}{3}}(t)L_0^{\frac{2}{3}}(\Phi^{-1}(\alpha_0 I_2(t)))}\,, \tag{3}$$

where $L_0 : \Delta_{Y_0} \to]0, +\infty[$ – is continuously differentiable and slowly varying function as $y \to Y_0$ with properties (2), *then it is sufficient that*

$$\lim_{t\uparrow\omega} \pi_\omega(t) p^{\frac{1}{3}}(t) L^{\frac{1}{3}}\big(\Phi^{-1}(\alpha_0 I_2(t))\big) = \infty, \quad \alpha_0 \lim_{t\uparrow\omega} I_2(t) = Z \tag{4}$$

and the following inequalities

$$\alpha_0\mu_0\mu_1 > 0, \quad \mu^* I_2(t) > 0 \ \text{ when } \ t \in]a, \omega[\tag{5}$$

are satisfied, where $\mu^* = \mu_0\mu_1 \operatorname{sign} \Phi(y)$ when $y \in \Delta_{Y_0}$. Moreover, each of these solutions admits the following asymptotic representations

$$\begin{gathered} \Phi(y(t)) = \alpha_0 I_2(t)[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \\ \frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \alpha_0 p^{\frac{1}{3}}(t) L^{\frac{1}{3}}\big(\Phi^{-1}(\alpha_0 I_2(t))\big)[1 + o(1)] \ \ (k = 1, 2) \ \text{ as } \ t \uparrow \omega. \end{gathered} \tag{6}$$

If conditions (4), (5) *are satisfied and there is the finite or equal* $\pm\infty$ *limit* (3), *then for* $\alpha_0 = 1$ *there exists a three-parameter family of* $P_\omega(Y_0, 1)$*-solutions with the asymptotic representations* (6) *in the case when* $\mu^* > 0$ *and a two-parameter family in the case when* $\mu^* < 0$, *and for* $\alpha_0 = -1$ *– a one-parameter family of such solutions in the case when* $\mu^* > 0$.

The next three theorems are devoted to the cases when $\lambda_0 = \pm\infty$, $\lambda_0 = 0$. They are established on the condition that slowly varying function L at $y \to Y_0$ satisfies the S conditions.

Definition 2. The slowly varying as $y \to Y$ function $L : \Delta_Y \to]0, +\infty[$, where Y is equal to either zero or $\pm\infty$, and Δ_Y is a one-sided neighborhood of Y satisfies the S, if

$$L\big(\mu e^{[1+o(1)]\ln|y|}\big) = L(y)[1 + o(1)] \ \text{ as } \ y \to Y \ \ (y \in \Delta_Y),$$

where $\mu = \operatorname{sign} y$.

Theorem 2. *Let L satisfy S. Then for the existence of $P_\omega(Y_0, \pm\infty)$-solutions of equation* (1) *it is necessary and sufficient that*

$$\mu_0\mu_1\pi_\omega(t) > 0 \ \textit{ when } \ t \in]a, \omega[\,, \quad \mu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0, \tag{7}$$

$$\lim_{t\uparrow\omega} p(t)\pi_\omega^3(t)L(\mu_0\pi_\omega^2(t)) = 0, \quad \int\limits_{a_1}^{\omega} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau = +\infty, \tag{8}$$

where $a_1 \in [a, \omega[$ is such that $\mu_0\pi_\omega^2(t) \in \Delta_{Y_0}$ when $t \in [a_1, \omega[$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)| = 2\ln|\pi_\omega(t)| + \frac{\alpha_0}{2}\int\limits_{a_1}^{t} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau\,[1 + o(1)] \ \textit{ as } \ t \uparrow \omega, \tag{9}$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{3-k}{\pi_\omega(t)}\,[1 + o(1)] \ \ (k = 1, 2) \ \textit{ as } \ t \uparrow \omega. \tag{10}$$

If conditions (7), (8) *are satisfied, then there is a three-parameter family of* $P_\omega(Y_0, \pm\infty)$*-solutions with the asymptotic representations* (9), (10) *in the case of* $\omega = +\infty$, *and a one-parametric family of these solutions with the same representations when* $\omega < +\infty$.

Theorem 3. *Let L satisfy S and conditions* (7), (8) *hold. In addition, let the function $p : [a, \omega[\to]0, +\infty[$ be continuous and differentiable and such that there is a finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega p'(t)}{p(t)}\,.$$

Then for each $P_\omega(Y_0, \pm\infty)$-solutions of the differential equation (1) *the place asymptotic representations*

$$\ln|y(t)| = 2\ln|\pi_\omega(t)| + \frac{\alpha_0}{2}\int\limits_{a_1}^{t} p(\tau)\pi_\omega^2(\tau)L(\mu_0\pi_\omega^2(\tau))\,d\tau\,[1 + o(1)] \ \textit{ as } \ t \uparrow \omega,$$

$$\frac{y^{(k)}(t)}{y^{(k-1)}(t)} = \frac{1}{\pi_\omega(t)}\Big[3 - k + \frac{\alpha_0}{2}\,p(\tau)\pi_\omega^3(\tau)L(\mu_0\pi_\omega^2(\tau))[1 + o(1)]\Big] \ \ (k = 1, 2) \ \textit{ as } \ t \uparrow \omega$$

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Theorem 4. *Let L satisfy S. Then for the existence of $P_\omega(Y_0,0)$-solutions of equation* (1) *for which there is a finite or equal $\pm\infty$*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'''(t)}{y''(t)}\,,$$

it is necessary and sufficient that

$$\mu_0\mu_1\pi_\omega(t)>0, \ \ where\ \ t\in\,]a,\omega[\,, \quad \mu_0\lim_{t\uparrow\omega}|\pi_\omega(t)|=Y_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)p(t)}{I_1(t)}=-2, \tag{11}$$

$$\lim_{t\uparrow\omega}p(t)\pi_\omega^3(t)L(\mu_0|\pi_\omega(t)|)=0, \quad \int\limits_{a_1}^{\omega}p(\tau)\pi_\omega^2(\tau)L(\mu_0|\pi_\omega(\tau)|)\,d\tau=+\infty, \tag{12}$$

where $a_1\in[a,\omega[$ is such that $\mu_0|\pi_\omega(t)|\in\Delta_{Y_0}$ when $t\in[a_1,\omega[\,$. Moreover, each of solutions admits the following asymptotic representations

$$\ln|y(t)|=\ln|\pi_\omega(t)|-\alpha_0\int\limits_{a_1}^{t}p(\tau)\pi_\omega^2(\tau)L\big(\mu_0|\pi_\omega(\tau)|\big)\,d\tau\,[1+o(1)] \ \ as\ \ t\uparrow\omega, \tag{13}$$

$$\frac{y'(t)}{y(t)}=\frac{1+o(1)}{\pi_\omega(t)}\,, \quad \frac{y''(t)}{y'(t)}=-\alpha_0p(t)\pi_\omega^2(t)L\big(\mu_0|\pi_\omega(t)|\big)\,[1+o(1)] \ \ as\ \ t\uparrow\omega. \tag{14}$$

If conditions (11)*,* (12) *are satisfied, then there exists a two-parameter family of $P_\omega(Y_0,0)$-solutions with the asymptotic representations* (13)*,* (14)*.*

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