Weak Solutions for Coupled Stochastic Functional-Differential Equations in Infinite-Dimensional Spaces

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In this paper, we are concerned with the existence and uniqueness of weak and strong solutions to stochastic functional differential equations in a Hilbert space of the form

$$\begin{cases} du(t)=\big[Au(t)+f(u_t,y_t)\big]\,dt+\sigma(u_t,y_t)\,dW(t),\\ dy(t)=g(u_t,y_t)\,dt, \ \ t\geq 0,\\ u(t)=\phi(t), \quad y(t)=\psi(t), \ \ t\in[-h,0], \ \ h>0. \end{cases} \tag{0.1}$$

Here $u_t=u(t+\theta)$, $y_t=y(t+\theta)$, $\theta\in[-h,0]$, A is an infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t):\ t\geq 0\}$ in a separable Hilbert space H, $W(t)$ is a Q-Wiener process on a separable Hilbert space K, $u(t)$ is a state process, the functionals f and g map the space of functions continuous on $[-h,0]$ into H, σ maps the same space into a special space of Hilbert–Schmidt operators. Finally, $\phi,\psi:[-h,0]\to H$ are the initial condition functions.

Functional differential equations are mathematical models of processes whose evolution depends on their previous states. The paired stochastic equations of type (0.1) arise in various applications; for instance, the bidomain equation (defibrillator model), the Hodgkin–Huxley equation for nerve axons, the nuclear reactor dynamics equation, etc. These equations are characterized by the fact that one of them is a partial differential equation (infinite-dimensional), and the other is an ordinary one (finite-dimensional). The nonlinearities in such equations do not satisfy the Lipschitz condition, which complicates the proof of the existence and uniqueness. However, as a rule, the right-hand sides of these equations satisfy some monotonicity conditions, which makes it possible to apply Galerkin approximations. This method is the main technique for obtaining the existence and uniqueness of weak solutions in this paper.

1 Preliminaries and main results

Let K and H be two separable Hilbert spaces and let $V \subset H$ be a reflexive Banach space with the dual space H'. By identifying H with its dual H', we have $V \subset H \cong H' \subset V'$, where the inclusions are assumed to be continuous and dense. (V, H, V') is called a Gelfand triple. Let the norms in V, H and V' be denoted by $\|\cdot\|_V$, $\|\cdot\|$ and $\|\cdot\|_{V'}$, respectively. The inner product in H and the duality scalar product between V and V' will be denoted by $(\cdot\,,\cdot)$, and $\langle\cdot\,,\cdot\rangle$. The norm and inner product in K will be denoted by $\|\cdot\|_K$ and $(\cdot\,,\cdot)$, respectively.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with the normal filtration $\{\mathcal{F}_t:\ t \geq 0\}$ generated by the Q-Wiener process W on $(\Omega, \mathcal{F}, P)$ with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}$ in K and a sequence of nonnegative real numbers λ_k such that

$$Qe_k = \lambda_k e_k, \quad k = 1, 2, \dots, \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \beta_k(t) e_k,$$

where $\beta_k(t)$ are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$.

Let $U_0 = Q^{\frac{1}{2}}(U)$ and $L_2^0 = L_2(U_0, H)$ be the space of all Hilbert-Schmidt operators from U_0 to H with the inner product $(\Phi, \Psi)_{L_2^0} = \operatorname{tr}[\Phi Q \Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively. $C := C([-h, 0]; H)$ is the space of continuous mappings from $[-h, 0]$ to H equipped with the norm $\|u\|_C = \sup\limits_{[-h,0]} \|u(\theta)\|$, and $L_V^2 := L^2((-h, 0); V)$ is the space of V-valued mappings with the norm

$$\|u\|_{L_V^2}^2 = \int_{-h}^{0} \|u(t)\|_V^2\, dt.$$

We impose the following **conditions on the operator A**:

(A1) A is a linear operator with domain $D(A)$ dense in H such that $A : V \to V'$.

(A2) For any $u, v \in V$ there exists $\alpha > 0$ such that

$$|\langle Au, v\rangle| \leq \alpha \|u\|_V \cdot \|v\|_V.$$

(A3) A satisfies the coercivity condition: there exist constants $\beta > 0$ and γ such that

$$\langle Av, v\rangle \leq -\beta \|v\|_V^2 + \gamma \|v\|^2, \quad \forall\, v \in V.$$

Conditions on nonlinearities:

(N1) f and g are mappings from $C \cap L_V^2 \times C$ to H, and σ is a mapping from $C \cap L_V^2 \times C$ to L_2^0.

(N2) (Growth condition) There exist positive constants $\alpha > 0$ and $\gamma \geq 1$ such that

$$\|f(\phi, \psi)\| + \|g(\phi, \psi)\| \leq \alpha \bigg(1 + \Big(\int_{-h}^{0} \|\phi\|_V\, dt\Big)^{\gamma} + \|\phi\|_C^{\gamma} + \|\psi\|_C^{\gamma}\bigg)$$

and

$$\|\sigma(\phi, \psi)\|_{L_2^0}^2 \leq \alpha\big(1 + \|\phi\|_C^2 + \|\psi\|_C^2\big).$$

(N3) (Local Lipschitz condition) For any $N > 0$ there exists a constant $K_N > 0$ such that

$$\|f(\phi,\psi) - f(\phi_1,\psi_1)\|^2 + \|g(\phi,\psi) - g(\phi_1,\psi_1)\|^2 + \|\sigma(\phi,\psi) - \sigma(\phi_1,\psi_1)\|^2_{L_2^0} \\ \leq K_N\big(\|\phi - \phi_1\|^2_C + \|\psi - \psi_1\|^2_C\big)$$

for any $\phi, \phi_1 \in C \cap L^2_V$ and $\psi, \psi_1 \in C$ with $\|\phi\|^2_C + \|\psi\|^2_C < N$, $\|\phi_1\|^2_C + \|\psi_1\|^2_C < N$.

(N4) (Coercivity condition) There exist constants $\beta > 0$, λ and C_1 such that

$$\langle A\phi(0), \phi(0)\rangle + (f(\phi,\psi), \phi(0)) + (g(\phi,\psi), \psi(0)) + \|\sigma(\phi,\psi)\|^2_{L_2^0} \\ \leq -\beta\|\phi(0)\|^2_V + \lambda\big(\|\phi\|^2_C + \|\psi\|^2_C\big) + C_1.$$

(N5) (Monotonicity condition) For any $\phi, \phi_1 \in C \cap L^2_V$ and $\psi, \psi_1 \in C$, we have

$$2\big\langle A(\phi(0) - \phi_1(0), \phi(0) - \phi_1(0))\big\rangle + 2\big(f(\phi,\psi) - f(\phi_1,\psi_1), \phi(0) - \phi_1(0)\big) \\ + 2\big(g(\phi,\psi) - g(\phi_1,\psi_1), \psi(0) - \psi_1(0)\big) + \big\|\sigma(\phi,\psi) - \sigma(\phi_1,\psi_1)\big\|^2_{L_2^0} \\ \leq \delta\big(\|\phi - \phi_1\|^2_C + \|\psi - \psi_1\|^2_C\big)$$

for some constant $\delta > 0$.

Let $\phi(t) \in C \cap L^2_V$ and $\psi(t) \in C$, $t \in [-h, 0]$.
Let $\Omega_T = [0,T] \times \Omega$.

Definition. We call an $\mathcal{F}_t$-adapted random process $(u(t), y(t)) \in V \times H$ a weak solution of the initial problem (0.1) on $[0,T]$ if:

(1) $u(t) = \phi(t),\ y(t) = \psi(t),\ t \in [-h, 0]$;

(2) $u \in L^2(\Omega_T, V),\ y \in L^2(\Omega_T, H)$;

(3) for any $v \in V$ and $z \in H$, the equations

$$(u(t), v) = (u(0), v) + \int_0^t \big(\langle Au(s), v\rangle + (f(u_s, y_s), v)\big)\,ds + \int_0^t (\sigma(u_s, y_s)\,dW(s), v),$$

$$(y(t), z) = (y(0), z) + \int_0^t (g(u_s, y_s), z)\,dz$$

hold a.s. for each $t \in [0,T]$.

Theorem 1.1 (Existence and uniqueness)**.** *Suppose that conditions* (A1)–(A3) *and* (N1)–(N5) *hold. Then, for every $\phi \in C \cap L^2_V$ and $\psi \in C$, the initial problem* (0.1) *has a unique weak solution $(u(t), y(t))$ on $[0,T]$ such that*

$$u \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega_T, V),\ \ y \in L^2(\Omega, C([0,T]; H)).$$

Moreover, the energy equation holds:

$$\|u(t)\|^2 + \|y(t)\|^2 = \|u(0)\|^2 + \|y(0)\|^2 \\ + 2\int_0^t \Big(\langle Au(s), u(s)\rangle + (f(u_s, y_s), u(s)) + (g(u_s, y_s), y(s))\Big)\,ds \\ + \int_0^t \|\sigma(u_s, y_s)\|^2_{L_2^0}\,ds + 2\int_0^t \big(\sigma(u_s, y_s)\,dW(s), u(s)\big). \quad (1.1)$$

2 Proof of the main result

In this section, we provide the sketch of the proof of Theorem 1.1.

Proof.
Uniqueness. Suppose that $(u(t),y(t))$ and $(u^1(t),y^1(t))$ are two weak solutions of the initial problem (0.1). Then, in view of (1.1) and condition (N5), we can easily show that

$$\mathbf{E}\|u(t)-u^1(t)\|^2+\mathbf{E}\|y(t)-y^1(t)\|^2=2\mathbf{E}\int\limits_0^t\big\langle A(u(s)-u^1(s)),u(s)-u^1(s)\big\rangle\,ds$$
$$+2\mathbf{E}\int\limits_0^t\Big[\big(f(u_s,y_s)-f(u_s^1,y_s^1),u(s)-u^1(s)\big)+\big(g(u_s,y_s)-g(u_s^1,y_s^1),y(s)-y^1(s)\big)\Big]\,ds$$
$$+\mathbf{E}\int\limits_0^t\big\|\sigma(u_s,y_s)-\sigma(u_s^1,y_s^1)\big\|_{L_2^0}^2\,ds\le\delta\mathbf{E}\int\limits_0^t\big(\|u_s-u_s^1\|_C^2+\|y_s-y_s^1\|_C^2\big)\,ds.\quad(2.1)$$

In what follows, we will need the following obvious statement.

Lemma. *The following inequality holds:*

$$\mathbf{E}\sup_{t\in[0,T]}\big(\|u_t\|_C^2+\|y_t\|_C^2\big)\le\mathbf{E}\big(\|\phi\|_C^2+\|\psi\|_C^2\big)+\mathbf{E}\sup_{t\in[0,T]}\big(\|u(t)\|^2+\|y(t)\|^2\big).\quad(2.2)$$

So, taking into account (2.2), from (2.1) we obtain

$$\sup_{s\in[0,T]}\mathbf{E}\big(\|u(s)-u^1(s)\|^2+\|y(s)-y^1(s)\|^2\big)\le\delta\int\limits_0^t\sup_{\tau\in[0,s]}\mathbf{E}\big(\|u_\tau-u_\tau^1\|_C^2+\|y_\tau-y_\tau^1\|_C^2\big)\,ds$$
$$\le\delta\int\limits_0^t\sup_{\tau\in[0,s]}\mathbf{E}\big(\|u(\tau)-u^1(\tau)\|^2+\|y(\tau)-y^1(\tau)\|^2\big)\,ds,$$

which, by Gronwall's inequality, yields

$$\mathbf{E}\big(\|u(s)-u^1(s)\|^2+\|y(s)-y^1(s)\|^2\big)=0,\ \ \forall\,t\in[0,T],$$

which establishes the uniqueness.

Existence. We will prove the existence by using Galerkin approximations.

Step 1. Finite-dimensional case. Approximate solutions.

Let $\{v_k\}$ be a complete orthonormal basis for H with $v_k\in V$, and let $H_n=\operatorname{span}\{v_1,\dots,v_n\}$. Suppose that $P_n:H\to H_n$ is an orthogonal projector such that

$$P_nh=\sum_{k=1}^n(h,v_k)v_k\ \text{ for }\ h\in H.$$

We extend P_n to the projection operator $P_n':V'\to V_n'$ defined as

$$P_n'w=\sum_{k=1}^n\langle w,v_k\rangle v_k\ \text{ for }\ w\in V'.$$

Obviously, $V_n = H_n = V_n'$.

Let $K_n = \operatorname{span}\{e_1, \dots, e_n\}$. We denote by Π_n a projection operator from K to K_n such that

$$\Pi_n a = \sum_{k=1}^{n} (a, e_k) e_k.$$

Let us introduce the following notation:

$$A^n u = P_n' A u, \quad f^n(\phi, \psi) = P_n f(\phi, \psi), \quad g^n(\phi, \psi) = P_n g(\phi, \psi), \quad \sigma^n(\phi, \psi) = P^n \sigma(\phi, \psi),$$

for $u \in V, \phi \in C \cap L_V^2$, and $\psi \in C$.

We consider the approximate equations to equations (0.1):

$$\begin{aligned} du^n(t) &= \big[A^n u^n(t) + f^n(u_t^n, y_t^n)\big]\, dt + \sigma^n(u_t^n, y_t^n)\, dW^n(t),\\ dy^n(t) &= g^n(u_t^n, y_t^n)\, dt,\\ &\quad u^n(t) = P_n \phi(t), \quad y^n(t) = P_n \psi(t), \ \ t \in [-h, 0], \end{aligned} \tag{2.3}$$

for $t \in [0, T]$, where $W^n(t) = \Pi_n W(t)$.

The above equations can be regarded as Itô equations in $\mathbb{R}^n$. It can be shown that, under conditions (N1)–(N5), the coefficients f^n, σ^n and g^n of these equations are locally bounded and Lipschitz continuous and monotone. Hence, (2.3) has a unique solution $(u^n(t), y^n(t))$ in V_n on any finite time interval $[0, T]$. Moreover, it satisfies the property $u^n \in L^2(\Omega, C([0, T]; H)) \cap L^2(\Omega_T, V)$ and $y^n \in L^2(\Omega, C([0, T]; H))$.

Step 2. A priori estimate.

Next, we will establish a priori estimates with some positive constant A:

$$\mathbf{E} \sup_{t \in [0,T]} \big(\|u^n(t)\|^2 + \|y^n(t)\|^2\big) + \int_0^T \mathbf{E}\|u^n(s)\|_V^2 \, ds \le A. \tag{2.4}$$

Step 3. Weak limits.

It follows from (2.4) that there exists subsequences, denoted for convenience by u^n and y^n such that $u^n \to u$ weakly in $L^2(\Omega_T, V)$ and $y^n \to y$ weakly in $L^2(\Omega_T, V)$. Next, we justify the passage to the limit in the finite-dimensional equation, which proves the theorem. $\square$

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