Boundary Value Problems for Implicit Fractional Differential Equations at Resonance

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1 Introduction

Let $T > 0$ be given, $J = [0, T]$, and $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$.

We discuss the implicit fractional differential equation

$$^c\!D^{\alpha}u(t) = a(t)\,{}^c\!D^{\beta}u(t) + f\big(t, u(t), {}^c\!D^{\alpha}u(t)\big) \tag{1.1}$$

together with the boundary condition

$$u(0) = \sum_{k=1}^{n} c_k u(\rho_k), \tag{1.2}$$

where $0 < \beta < \alpha < 1$, $a \in C(J)$, $f \in C(J \times \mathbb{R}^2)$, ${}^c\!D$ denotes the Caputo fractional derivative and $n \in \mathbb{N}$, $0 < \rho_1 < \rho_2 < \cdots < \rho_n \le T$, $c_k > 0$, $\sum\limits_{k=1}^{n} c_k = 1$. Further conditions for a, f will be specified later.

Definition 1.1. We say that $u : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $u, {}^c\!D^{\alpha}u \in C(J)$ and (1.1) holds for $t \in J$. A solution u of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

We recall the definitions of the Caputo fractional derivative and the Riemann–Liouville fractional integral [1, 2]. The *Caputo fractional derivative* ${}^c\!D^{\gamma}x$ of order $\gamma \in (0, 1)$ of a function $x : J \to \mathbb{R}$ is given as

$${}^c\!D^{\gamma}x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s) - x(0))\,\mathrm{d}s,$$

and the *Riemann–Liouville fractional integral* $I^{\gamma}x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma}x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function. It is not difficult to verify that

$$0 < \gamma < \mu < 1, \quad x, {}^c\!D^{\mu}x \in C(J) \Longrightarrow {}^c\!D^{\gamma}x(t) = I^{\mu-\gamma}\,{}^c\!D^{\mu}x(t), \ \ t \in J. \tag{1.3}$$

Remark 1.1. It follows from (1.3) that u is a solution of (1.1) if and only if it is a solution of the implicit equation
$$ {}^c\!D^\alpha u(t) = a(t) I^{\alpha-\beta}\, {}^c\!D^\alpha u(t) + f\big(t, u(t), {}^c\!D^\alpha u(t)\big). $$

We note that for $n = 1$, $c_1 = 1$ and $\rho_1 = T$, the boundary condition (1.2) reduces to the periodic condition $x(0) = x(T)$.

Problem (1.1), (1.2) is at resonance, because each constant function u on J is a solution of problem ${}^c\!D^\alpha u = a(t)\,{}^c\!D^\beta u$, (1.2).

We suppose that the functions a, f satisfy the conditions:

(H_1) $a(t) \le 0$ for $t \in J$.

(H_2) There exist $D, H \in \mathbb{R}$, $D < H$, such that
$$ \begin{aligned} f(t, D, y) > 0 \ \text{ for } \ t \in J, \ \ y \le 0, \\ f(t, H, y) < 0 \ \text{ for } \ t \in J, \ \ y \ge 0. \end{aligned} $$

(H_3) There exists $L > 0$ such that
$$ \big| f(t, x, y_1) - f(t, x, y_2) \big| \le L |y_1 - y_2| \ \text{ for } \ t \in J, \ \ x \in [C, D], \ \ y_1, y_2 \in \mathbb{R}, $$
and
$$ Q = \frac{\|a\| T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + L < 1. $$

The aim of this paper is to study the existence of solutions to problem (1.1), (1.2). The existence result is proved by the initial value method. To this end we first introduce an operator $\mathcal{F} : C(J) \to C(J)$. We prove that for each $c \in [D, H]$ the initial value problem ${}^c\!D^\alpha x(t) = \mathcal{F}x(t)$, $x(0) = c$ has a solution on J and all its solutions x satisfy $D < x < H$ on $(0, T]$. Let $\mathcal{C}$ be the set of all solutions to this problem for $c \in [D, H]$. We prove that there exists at least one $u \in \mathcal{C}$ satisfying (1.2) and u is a solution of problem (1.1), (1.2).

2 Operator $\mathcal{F}$ and its properties

Let $f^* : J \times \mathbb{R}^2$ be defined as
$$ f^*(t, x, y) = \begin{cases} f(t, H, y) & \text{if } \ x > H, \\ f(t, x, y) & \text{if } \ x \in [D, H], \\ f(t, D, y) & \text{if } \ x < D \end{cases} $$
for $t \in J$ and $y \in \mathbb{R}$. Then $f^* \in C(J \times \mathbb{R}^2)$,
$$ \begin{aligned} f^*(t, x, y) > 0 \ \text{ if } \ t \in J, \ \ x \le D, \ \ y \le 0, \\ f^*(t, x, y) < 0 \ \text{ if } \ t \in J, \ \ x \ge H, \ \ y \ge 0, \end{aligned} \tag{2.1} $$
$$ \big| f^*(t, x, y_1) - f^*(t, x, y_2) \big| \le L |y_1 - y_2|, \ \ t \in J, \ \ x, y_1, y_2 \in \mathbb{R}, \tag{2.2} $$
and
$$ |f^*(t, x, 0)| \le M, \ \ t \in J, \ \ x \in \mathbb{R}, \tag{2.3} $$
where
$$ M = \max\big\{ |f(t, x, 0)| : \ t \in J, \ x \in [D, H] \big\}. $$

The following result is proved by the Banach fixed point theorem.

Lemma 2.1. *Let $x \in C(J)$. Then there exists a unique solution $w \in C(J)$ of the equation*

$$w(t) = a(t)I^{\alpha-\beta}w(t) + f^*(t, x(t), w(t)). \tag{2.4}$$

Keeping in mind Lemma 2.1, for each $x \in C(J)$ there exists a unique solution $w \in C(J)$ of equation (2.4). We put $w = \mathcal{F}x$ and have an operator $\mathcal{F} : C(J) \to C(J)$ satisfying the equality

$$\mathcal{F}x(t) = a(t)I^{\alpha-\beta}\mathcal{F}x(t) + f^*(t, x(t), \mathcal{F}x(t)) \ \text{ for } \ t \in J, \ \ x \in C(J). \tag{2.5}$$

The properties of $\mathcal{F}$ are given in the following two results.

Lemma 2.2. *$\mathcal{F} : C(J) \to C(J)$ is a continuous operator and*

$$\|\mathcal{F}x\| \le \frac{M}{1-Q}\,, \ \ x \in C(J),$$

where M is from (2.3) *and Q from* (H_3).

Lemma 2.3. *If $u \in C(J)$ is a solution of the equation*

$${}^{c}D^{\alpha}u(t) = \mathcal{F}u(t), \tag{2.6}$$

then u is a solution of the equation

$${}^{c}D^{\alpha}u(t) = a(t){}^{c}D^{\beta}u(t) + f^*\big(t, u(t), {}^{c}D^{\alpha}u(t)\big). \tag{2.7}$$

Proof. Let $u \in C(J)$ be a solution of (2.6). Then $\mathcal{F}u \in C(J)$ and so ${}^{c}D^{\alpha}u \in C(J)$. Hence, by (2.5) and (1.3) (see Remark 1.1),

$$\begin{aligned} {}^{c}D^{\alpha}u &= a(t)I^{\alpha-\beta}\mathcal{F}u + f^*(t, u, \mathcal{F}u) \\ &= a(t)I^{\alpha-\beta}{}^{c}D^{\alpha}u + f^*(t, u, {}^{c}D^{\alpha}u) = a(t){}^{c}D^{\beta}u + f^*(t, u, {}^{c}D^{\alpha}u). \end{aligned}$$

As a result u is a solution of (2.7). □

3 Initial value problem

We investigate the initial value problem

$${}^{c}D^{\alpha}u(t) = \mathcal{F}u(t), \tag{3.1}$$

$$u(0) = c, \tag{3.2}$$

where $c \in \mathbb{R}$. It is easy to check that $u \in C(J)$ is a solution of problem (3.1), (3.2) if and only if u is a fixed point of the operator $\mathcal{L}_c : C(J) \to C(J)$, $\mathcal{L}_c x(t) = c + I^{\alpha}\mathcal{F}x(t)$.

The following existence result is proved by the Schauder fixed point theorem.

Lemma 3.1. *Let $c \in \mathbb{R}$. Then problem* (3.1), (3.2) *has at least one solution.*

For $c \in \mathbb{R}$, let $\mathcal{S}(c)$ be the set of all solutions to problem (3.1), (3.2). By Lemma 3.1, $\mathcal{S}(c) \neq \varnothing$.

Lemma 3.2. *Let $c \in [D, H]$ and $x \in \mathcal{S}(c)$. Then*

$$D < x(t) < H \ \ \textit{for} \ \ t \in (0, T]. \tag{3.3}$$

Proof. By Lemma 2.3, ${}^c\!D^\alpha x(t) = a(t){}^c\!D^\beta x(t) + f^*(t, x(t), {}^c\!D^\alpha x(t))$, $t \in J$. Suppose that $\max\{x(t): t \in J\} = x(\xi) \geq H$ for some $\xi \in (0, T]$. Then it follows from the maximum principle for the Caputo fractional derivative [3, Lemma 2.1] that

$$ {}^c\!D^\gamma x(t)\big|_{t=\xi} \geq \frac{x(\xi) - x(0)}{\Gamma(1-\gamma)}\, \xi^{-\gamma} = \frac{x(\xi) - c}{\Gamma(1-\gamma)}\, \xi^{-\gamma} \geq 0 \ \text{ for } \ \gamma \in (0,1). $$

Hence (see (H_1) and (2.1)),

$$ a(\xi){}^c\!D^\beta x(t)\big|_{t=\xi} + f^*\big(\xi, x(\xi), {}^c\!D^\alpha x(t)\big|_{t=\xi}\big) < 0, $$

contrary to ${}^c\!D^\alpha x(t)|_{t=\xi} \geq 0$ and

$$ {}^c\!D^\alpha x(t)\big|_{t=\xi} = a(\xi){}^c\!D^\beta x(t)\big|_{t=\xi} + f^*\big(\xi, u(\xi), {}^c\!D^\alpha x(t)\big|_{t=\xi}\big). $$

Therefore $x < H$ on $(0, T]$. Similarly, for $x > D$ on this interval. □

4 Problem (1.1), (1.2)

Let

$$ \mathcal{C} = \bigcup_{c \in [D,H]} \mathcal{S}(c). $$

Then $\mathcal{C} \neq \varnothing$, $\mathcal{C}$ is a compact subset in $C(J)$ and, by Lemma 3.2,

$$ D < x(t) < H \ \text{ for } \ t \in (0, T], \ \ x \in \mathcal{C}. \tag{4.1} $$

Theorem 4.1. *Let* (H$_1$)–(H$_3$) *hold. Then problem* (1.1), (1.2) *has at least one solution u and $D < u(t) < H$ for $t \in (0, T]$.*

Proof. Suppose that

$$ x(0) \neq \sum_{k=1}^{n} c_k x(\rho_k) \ \text{ for } \ x \in \mathcal{C}. \tag{4.2} $$

Let

$$ \mathcal{C}_c^+ = \Big\{ x \in \mathcal{S}(c): \ x(0) < \sum_{k=1}^{n} c_k x(\rho_k) \Big\}, $$
$$ \chi^+ = \big\{ c \in [D, H]: \ \mathcal{C}_c^+ \neq \varnothing \big\}, \quad \chi^- = [D, H] \setminus \chi^+. $$

We observe that if $c \in \chi^-$ and $x \in \mathcal{S}(c)$, then $x(0) > \sum\limits_{k=1}^{n} c_k x(\rho_k)$. Since, by (4.1), $\mathcal{S}(D) = \mathcal{C}_D^+$ and $\mathcal{C}_H^+ = \varnothing$, we have $D \in \chi^+$ and $H \in \chi^-$. Hence χ^+ and χ^- are nonempty sets. We can prove that χ^+ and χ^- are closed in $[D, H]$. Hence the compact interval $[D, H]$ is the union of two nonempty, closed and disjoint subsets χ^+, χ^-, which is impossible. Thus assumption (4.2) is false, and therefore there exists $u \in \mathcal{C}$ such that $u(0) = \sum\limits_{k=1}^{n} c_k u(\rho_k)$. Since, by (4.1), $D < u < H$ on $(0, T]$, we have

$$ f^*\big(t, u(t), {}^c\!D^\alpha u(t)\big) = f\big(t, u(t), {}^c\!D^\alpha u(t)\big) $$

for $t \in J$. As a result u is a solution of problem (1.1), (1.2). □

Example 4.1. Let $T=1$, $n\in\mathbb{N}$, $r\in C(J)$, $a(t)=t(t-1)$ and

$$f(t,x,y)=r(t)-x^{2n-1}+\sin x-(1/2)\arctan y.$$

Then $\|a\|=1/4$ and the functions a,f satisfy (H_1)–(H_3) for $H>\sqrt[2n-1]{\|r\|+1}$, $D=-H$ and $L=1/2$ since $\Gamma(v)>4/5$ for $v\in[1,2]$. By Theorem 4.1, there exists a solution u of the equation

$${}^{c}D^{\alpha}u=t(t-1)\,{}^{c}D^{\beta}u+r(t)-u^{2n-1}+\sin u-\frac{1}{2}\arctan{}^{c}D^{\alpha}u,$$

satisfying the boundary condition (1.2) and $D<u<H$ on $(0,1]$.

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