On Stability and Asymptotic Properties of Solutions of Second-Order Damped Linear Differential Equations with Periodic Non-Constant Coefficients

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Consider the equation

$$\boxed{x'' = p(t)x + g(t)x',} \tag{1}$$

where $p, g : \mathbb{R} \to \mathbb{R}$ are ω-periodic locally Lebesgue integrable functions, $\omega > 0$. By a solution to equation (1), as usual, we understand a function $x : \mathbb{R} \to \mathbb{R}$ which is locally absolutely continuous together with its first derivative and satisfies (1) almost everywhere in $\mathbb{R}$.

We first introduce the following definitions.

Definition 1. We say that the pair (p, g) belongs to the set $\mathcal{V}^-(\omega)$ (resp. $\mathcal{V}^+(\omega)$) if, for any function $u : [0, \omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative and satisfies

$$u''(t) \geq p(t)u(t) + g(t)u'(t) \quad \text{for a.e. } t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) \geq u'(\omega),$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [0, \omega] \quad \big(\text{resp. } u(t) \geq 0 \quad \text{for } t \in [0, \omega]\big)$$

holds.

Remark 1. In the related literature, the fact that $(p, g) \in \mathcal{V}^-(\omega)$ (resp. $(p, g) \in \mathcal{V}^+(\omega)$) is often called maximum (resp. anti-maximum) principle for the periodic problem

$$x'' = p(t)x + g(t)x'; \quad x(0) = x(\omega), \ x'(0) = x'(\omega). \tag{2}$$

Moreover, the relationship of the classes $\mathcal{V}^-(\omega)$ and $\mathcal{V}^+(\omega)$ with a sign of the Green's function of (2) is known.

Definition 2. We say that the pair (p, g) belongs to the set $\mathcal{V}_0(\omega)$ if the homogeneous problem (2) has a positive solution.

Definition 3. We say that the pair (p, g) belongs to the set $\mathcal{D}$ if any non-trivial solution to equation (1) has at most one zero in $\mathbb{R}$.

The aim of this note is not to provide conditions guaranteeing that the maximum (resp. anti-maximum) principle holds for (2). Let us mention only that such effective conditions are derived, e.g., in [1, 3, 5] (see, also [2, 4] for the case of $g(t) \equiv 0$).

Below we discuss the stability and asymptotic properties of solutions of equation (1), if the pair (p, g) of the coefficients in (1) belongs to each of the above-defined classes.

Theorem 2. *Let $(p,g) \in \mathcal{V}^-(\omega)$. Then, there exist $\mu_1, \mu_2 > 0$ and positive linearly independent solutions x_1, x_2 to equation* (1) *such that*

$$\mu_2 - \mu_1 = \frac{1}{\omega} \int\limits_0^\omega g(s)\,\mathrm{d}s$$

and

$$x_1(t) = \mathrm{e}^{-\mu_1 t}\,\varphi_1(t), \quad x_2(t) = \mathrm{e}^{\mu_2 t}\,\varphi_2(t) \quad \text{for } t \in \mathbb{R},$$

where $\varphi_1, \varphi_2 \in AC^1_{loc}(\mathbb{R})$ are ω-periodic functions; equation (1) *is unstable.*

Proposition 3. *Let $\int\limits_0^\omega g(s)\,\mathrm{d}s \geq 0$ and there exist a positive solution x to equation* (1) *satisfying*

$$x(t) = \mathrm{e}^{-\mu t}\,\varphi(t) \quad \text{for } t \in \mathbb{R},$$

where $\mu > 0$ and $\varphi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function. Then $(p,g) \in \mathcal{V}^-(\omega)$.

Proposition 4. *Let $\int\limits_0^\omega g(s)\,\mathrm{d}s \leq 0$ and there exist a positive solution y to equation* (1) *satisfying*

$$y(t) = \mathrm{e}^{\nu t}\,\psi(t) \quad \text{for } t \in \mathbb{R},$$

where $\nu > 0$ and $\psi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function. Then $(p,g) \in \mathcal{V}^-(\omega)$.

Following [4, Definition 13.1], we introduce the definition.

Definition 4. Equation (1) is said to be strongly exponential dichotomic, if there exist $\mu, \nu > 0$ and linearly independent solutions x, y to equation (1) such that the functions

$$t \mapsto \mathrm{e}^{\mu t}\,x(t), \quad t \mapsto \mathrm{e}^{-\nu t}\,y(t)$$

are positive and ω-periodic on $\mathbb{R}$.

Corollary 1. *Equation* (1) *is strongly exponential dichotomic if and only if $(p,g) \in \mathcal{V}^-(\omega)$.*

Theorem 5. *Let $(p,g) \in \mathcal{V}_0(\omega)$. Then, the following conclusions hold:*

(1) *If $\int\limits_0^\omega g(s)\,\mathrm{d}s > 0$, then equation* (1) *has linearly independent solutions x_1, x_2 such that x_1 is a positive ω-periodic solution and x_2 is a positive solution satisfying*

$$x(t) = \mathrm{e}^{\mu t}\,\varphi(t) \quad \text{for } t \in \mathbb{R},$$

where

$$\mu = \frac{1}{\omega} \int\limits_0^\omega g(s)\,\mathrm{d}s$$

and $\varphi \in AC^1_{loc}(\mathbb{R})$ is an ω-periodic function; equation (1) *is unstable.*

(2) *If $\int\limits_0^\omega g(s)\,\mathrm{d}s = 0$, then equation* (1) *has linearly independent solutions x_1, x_2 such that x_1 is a positive ω-periodic solution and x_2 is a solution, with exactly one zero in $\mathbb{R}$, satisfying*

$$\lim_{t \to -\infty} x_2(t) = -\infty, \quad \lim_{t \to +\infty} x_2(t) = +\infty;$$

equation (1) *is unstable.*

(3) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then equation* (1) *has linearly independent solutions* x_1, x_2 *such that* x_1 *is a positive* ω*-periodic solution and* x_2 *is a positive solution satisfying*

$$x(t) = \mathrm{e}^{-\mu t}\,\varphi(t) \quad \textit{for } t \in \mathbb{R},$$

where

$$\mu = -\frac{1}{\omega}\int\limits_0^\omega g(s)\,\mathrm{d}s$$

and $\varphi \in AC^1_{loc}(\mathbb{R})$ *is an* ω*-periodic function; equation* (1) *is stable.*

Theorem 6. *Let* $(p,g) \in \mathcal{V}^+(\omega)\cap\mathcal{D} = \operatorname{Int}\mathcal{V}^+(\omega)\cap\mathcal{D}$. *Then,* $\int\limits_0^\omega g(s)\,\mathrm{d}s \neq 0$ *and the following conclusions hold:*

(1) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s > 0$, *then equation* (1) *has a positive solution* x_0 *satisfying*

$$\lim_{t\to-\infty} x_0(t) = 0, \quad \lim_{t\to+\infty} x_0(t) = +\infty$$

and, moreover, every solution x *to equation* (1) *has at most one zeros in* $\mathbb{R}$ *and satisfies*

$$\lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to+\infty} |x(t)| = +\infty;$$

equation (1) *is unstable.*

(2) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then equation* (1) *has a positive solution* x_0 *satisfying*

$$\lim_{t\to-\infty} x_0(t) = +\infty, \quad \lim_{t\to+\infty} x_0(t) = 0$$

and, moreover, every solution x *to equation* (1) *has at most one zeros in* $\mathbb{R}$ *and satisfies*

$$\lim_{t\to-\infty} |x(t)| = +\infty, \quad \lim_{t\to+\infty} x(t) = 0;$$

equation (1) *is asymptotically stable.*

Theorem 7. *Let* $(p,g) \in \operatorname{Int}\mathcal{V}^+(\omega)\setminus\mathcal{D}$. *Then, every non-trivial solution to equation* (1) *is oscillatory in the neighbourhood* $+\infty$ *as well as* $-\infty$ *and the following conclusions hold:*

(1) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s > 0$, *then every non-trivial solution* x *to equation* (1) *satisfies*

$$\lim_{t\to-\infty} x(t) = 0, \quad \limsup_{t\to+\infty} x(t) = +\infty, \quad \liminf_{t\to+\infty} x(t) = -\infty;$$

equation (1) *is unstable.*

(2) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s = 0$, *then every solution to equation* (1) *is bounded; equation* (1) *is stable.*

(3) *If* $\int\limits_0^\omega g(s)\,\mathrm{d}s < 0$, *then every non-trivial solution* x *to equation* (1) *satisfies*

$$\limsup_{t\to-\infty} x(t) = +\infty, \quad \liminf_{t\to-\infty} x(t) = -\infty, \quad \lim_{t\to+\infty} x(t) = 0;$$

equation (1) *is asymptotically stable.*

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References

[1] I. V. Barteneva, A. Cabada and A. O. Ignatyev, Maximum and anti-maximum principles for the general operator of second order with variable coefficients. *Appl. Math. Comput.* **134** (2003), no. 1, 173–184.

[2] A. Cabada, J. Á. Cid and L. López-Somoza, *Maximum Principles for the Hill's Equation.* Academic Press, London, 2018.

[3] R. Hakl and P. J. Torres, Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign. *Appl. Math. Comput.* **217** (2011), no. 19, 7599–7611.

[4] A. Lomtatidze, Theorems on differential inequalities and periodic boundary value problem for second-order ordinary differential equations. *Mem. Differ. Equ. Math. Phys.* **67** (2016), 1–129.

[5] E. L. Tonkov, The second order periodic equation. (Russian) *Dokl. Akad. Nauk SSSR* **184** (1969), 296–299.