Cauchy's Problem for Singular Perturbed Systems of Differential Equations with Nonstable First-Order Turning Point

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A class of singularly perturbed differential equations (SPDE) with turning points is an effective model for the studies of various physical phenomena. There is a wide spectrum of papers devoted to the investigation of such problems and to the construction of the uniform asymptotic of the solution. This spectrum of SPDE is represented by R. Langer, W. Wasow, C. Lin, S. Lomov etc. Generalization on the class of systems of SPDE in the above-mentioned direction of the research is a relevant problem also nowadays.

In the paper [5], a system of SPDE with a stable turning point has been considered. In this case we have used the apparatus of Airy-Dorodnitsyn functions [1, 3]. An unstable turning point assumes a use of the following Airy-Langer functions:

$$TW = W'' \equiv W''(t) - tW(t) = 0.$$

Let us consider a system of SPDE with a stable turning point (SSPDE):

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x),$$

is a known matrix where

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-4, 0]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = col(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = col(0, 0, h(x))$ is a given vector function.

The system to be studied here (0.1) will be investigated under the following conditions:

(1) $\widetilde{a}(x), b(x), h(x) \in C^\infty[-4; 0]$;

(2) $a(x) \equiv x\widetilde{a}(x)$, $\widetilde{a}(x) = 3x$, $b(x) = 3x + 20$, $h(x) = 6x + 2$.

The scalar reduced equation for this matrix will be

$$x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x). \tag{0.2}$$

The analysis of such kind of problems and construction of uniform asymptotic solution on a given segment with a turning point brings certain difficulties and problems in the construction of asymptotic forms [3].

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$|A(x,0) - \lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are: $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}$.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions that appear in the solution of system (0.1) due to the special point

$$t = \mu^{-2} \cdot \varphi(x),$$

where $\mu = \varepsilon^{\frac{1}{3}}$, exponent p and regularizing function $\varphi(x)$ are to be determined. Instead of $Y(x,\varepsilon)$ function $\widetilde{Y}(x,t,\varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y_k}(x,t,\varepsilon)\big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y_k(x,\varepsilon),$$

which is the necessary condition for suggested method. The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y_k}(x,t,\varepsilon) \equiv \mu\varphi' \, \frac{\partial \widetilde{y}(x,t,\varepsilon)}{\partial t} + \mu^3 \, \frac{\partial \widetilde{y}(x,t,\varepsilon)}{\partial x} - A(x,\varepsilon)\widetilde{Y_k}(x,t,\varepsilon) = H(x). \tag{1.1}$$

We describe the space of functions in which it will be possible to construct a uniform asymptotic solution of the transformed system (1.1)

$$D_{1k} = \alpha_{1k}(x)\mathrm{Ai}(\mathrm{t}) + \beta_{1k}(x)\mathrm{Ai}'(\mathrm{t}), \quad D_{2k} = \alpha_{2k}(x)\mathrm{Bi}(\mathrm{t}) + \varepsilon^\gamma \beta_{2k}(x)\mathrm{Bi}'(\mathrm{t}),$$
$$D_{3k} = f_k(x)\nu(t) + \varepsilon^\gamma g_k(x)\nu'(t), \quad D_{4k} = \omega_k(x),$$

where $\alpha_{ik}(x), \beta_{ik}(x), f_k(x), g_k(x), \omega_k(x) \in C^\infty[-4,0]$.

Here functions Ai(t), Bi(t) are the Airy-Langer functions, $\nu(t)$ is an essentially special function [3].

The element of this space has the form

$$\widetilde{Y_k}(x,t,\varepsilon) = \sum_{i=1}^{2} \big[\alpha_{ik}(x)U_i(t) + \beta_{ik}(x)U_i'(t)\big] + f_k(x)\nu(t) + \varepsilon^\gamma g_k(x)\nu'(t) + \omega_k(x).$$

Denote the Airy-Langer functions as $U_1(t) \equiv \mathrm{Ai}(\mathrm{t})$, $U_2(t) \equiv \mathrm{Bi}(\mathrm{t})$.

Now we have to investigate how the transformed operator $\widetilde{L}_\varepsilon$ acts on the elements of the Space of non-resonant solutions D_{1k} and D_{2k}. Let us write the obtained result in the form of the following vector equations

$$\begin{aligned} U_i'(t) : \alpha_k(x,\varepsilon)\varphi'(x) - \big[A_0(x) + \mu^3 A_1\big]\beta_k(x,\varepsilon) = -\mu^3\beta_k'(x,\varepsilon), \\ U_i(t) : \beta_k(x,\varepsilon)\varphi(x)\varphi'(x) - \big[A_0(x) + \mu^3 A_1\big]\alpha_k(x,\varepsilon) = -\mu^3\alpha_k'(x,\varepsilon). \end{aligned} \tag{1.2}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.1) are sought as following vector function series ($i = 1, 2$):

$$\alpha_k(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \alpha_{kr}(x), \quad \beta_k(x,\varepsilon) = \sum_{r=0}^{+\infty} \mu^r \beta_{kr}(x). \tag{2.1}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi(x) = \bigg(\frac{3}{2}\int_0^x \sqrt{-x\widetilde{a}(x)}\,dx\bigg)^{\frac{2}{3}}.$$

The regularizing function of such kind has been considered in [2,4]. Due to such a choice of the regularizing variable $\det\Phi(x)\equiv 0$, there is a nontrivial solution of the homogeneous system (1.1) that is

$$Z_{k0}(x)=colon\Big(0,\frac{1}{\varphi'(x)}\,\beta_{i30}(x),-\varphi(x)\varphi'(x)\beta_{i20}(x),0,\beta_{i20}(x),\beta_{i30}(x)\Big), \tag{2.2}$$

where $\beta_{0ik}(x)$, $i=1,2$, $i=1,2,3$ are arbitrary up to some point and sufficiently smooth functions at $x\in[-4;0]$.

Two linearly independent solutions of the system (1.1) are

$$D_k(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r[\alpha_{ikr}(x)U_i(t)+\varepsilon^{\frac13}\beta_{ikr}(x,\varepsilon)U_i'(t)],\ \ i=1,2, \tag{2.3}$$

where $\alpha_{ikr}(x)=col(\alpha_{i1r}(x),\alpha_{i2r}(x),\alpha_{i3r}(x))$ and $\beta_{ikr}(x)=col(\beta_{i1r}(x),\beta_{i2r}(x),\beta_{i3r}(x))$ are known vector-functions.

Thus, gradual solving of systems of equations $t=\varepsilon^{-\frac23}\cdot\varphi(x)$, $i=1,2$, then gives two formal solutions of the transformation vector equation

$$D_k(x,\varepsilon^{-\frac23}\varphi(x),\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\Big[\alpha_{ikr}(x)U_i(\varepsilon^{-\frac23}\varphi(x))+\varepsilon^{\frac13}\beta_{ikr}(x,\varepsilon)U_i'(\varepsilon^{-\frac23}\varphi(x))\Big]. \tag{2.4}$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as the series

$$\omega(x,\varepsilon)\equiv\sum_{r=0}^{\infty}\varepsilon^r\omega_r(x)\equiv colon\Big(\sum_{r=0}^{\infty}\varepsilon^r\omega_{1r}(x),\sum_{r=0}^{\infty}\varepsilon^r\omega_{2r}(x),\sum_{r=0}^{\infty}\varepsilon^r\omega_{3r}(x)\Big). \tag{2.5}$$

3 Construction of formal partial solutions

To construct a partial solution of the SSPDE (0.1), let us analyze how transformation operator operates $\widetilde{L}_\varepsilon$ on an element from the space of non-resonant solutions D_{3r} and D_{4r}. The result is written in the form

$$\begin{aligned}&\widetilde{L}_\varepsilon(f_k(x,\varepsilon)\nu(t)+\mu g_k(x,\varepsilon)\nu'(t)+\omega_k(x,\varepsilon))\\&\quad=\mu f_k(x,\varepsilon)\varphi'(x)\nu(t)+g_k(x,\varepsilon)\varphi'(x)\varphi(x)\nu(t)-A(x,\varepsilon)f_k(x,\varepsilon)\nu(t)-\mu A(x,\varepsilon)g_k(x,\varepsilon)\nu'(t)\\&\qquad\quad+\mu^3f_k'(x)\nu(t)+\mu^4g_k'(x)\nu'(t)+\mu^2\varphi'(x)g_k(x)\pi^{-1}+\mu^3\omega'(x)-A(x,\varepsilon)\omega_k(x)=H(x).\end{aligned}$$

Therefore, the partial solution of the transformation vector equation (1.1) is then defined as the series

$$\widetilde{Y}_{\text{part.}}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\big[f_{kr}(x)\nu(t)+\varepsilon^{\frac13}g_{kr}(x)\nu'(t)\big]+\sum_{r=0}^{\infty}\varepsilon^r\bar{\omega}_{kr}(x).$$

Narrowing the solution, when $t=\varepsilon^{-\frac23}\cdot\varphi(x)$, the series

$$\widetilde{Y}_{\text{part.}}(x,t,\varepsilon)=\sum_{r=0}^{\infty}\varepsilon^r\bigg[f_{kr}(x)\nu(\varepsilon^{\frac23}\cdot\varphi(x))+\varepsilon^{\frac13}g_{kr}(x)\,\frac{d\nu(\varepsilon^{-\frac23}\cdot\varphi(x))}{d(\varepsilon^{-\frac23}\cdot\varphi(x))}\bigg]+\sum_{r=0}^{\infty}\varepsilon^r\bar{\omega}_{kr}(x), \tag{3.1}$$

is a formal partial solution of the SSPDE (0.1).

4 Estimation of the remainder terms of the asymptotic solution

In this paper we have considered the case of an unstable turning point. In this case the remainder terms of the solution have characteristic differences in comparison with the case of a stable turning point [3]. Let us write the formal solution of the transformed problem (1.1) in the following form:

$$\alpha_{ikr}(x,\varepsilon) \equiv \alpha_{ikr}(x,\varepsilon) + \varepsilon^{q+1}\xi_{\alpha(q+1)}(x,\varepsilon), \tag{4.1}$$

$$\beta_{ikr}(x,\varepsilon) \equiv \beta_{kr}(x,\varepsilon) + \varepsilon^{q+1}\xi_{\beta(q+1)}(x,\varepsilon), \tag{4.2}$$

where $\alpha_{kq}(x,\varepsilon)$ and $\beta_{kq}(x,\varepsilon)$ are partial q-sums of the series (1.1), $\varepsilon^{1+q}\xi_{\alpha(q+1)}(x,\varepsilon)$ and $\varepsilon^{1+q}\xi_{\beta(q+1)}(x,\varepsilon)$ are the remainder terms.

Let us write the main result of this paper in the following theorem.

Theorem. *Let for the SPDE system* (0.1) *the conditions* (1) *and* (2) *take place. Then for suffciciently small values of the parameter* $\varepsilon > 0$*:*

- *three linearly independent solutions of homogeneous transformed vector equation* (1.1) *can be built in form of series* (2.1) *and* (2.5)*;*
- *narrowing these solutions at* $t = \varepsilon^{-\frac{2}{3}} \cdot \varphi(x)$ *is the formal asymptotic solution of the homogeneous SPDE system* (0.1)*;*
- *partial solution of the nonhomogeneous SPDE system* (0.1) *constructed with the series* (3.1)*;*
- *for the remainder terms of the asymptotic solutions* (4.1), (4.2) *estimations are valid.*

References

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