On the Critical Case in the Theory of the Matrix Differential Equations

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In the mathematical description of various phenomena and processes arising in mathematical physics, electrical engineering, economics, have to deal with matrix differential equations. Therefore, such equations are relevant as for mathematicians and specialists in other fields of natural sciences. This article considers quasi-linear matrix differential equations with coefficients depicted in the form of absolutely and uniformly convergent Fourier series with slow variable in a sense coefficients and frequency (class F). The differences of the diagonal elements of the matrices of the linear part are pure imaginary, that is, we are dealing with a critical case. But between these diagonal elements assume certain relations that indicate the absence of resonance between the natural frequencies of the system and frequency of external excitation force. The problem is considered establishing signs of existence in such an equation of solutions class F. By means of a number of transformations the equation is reduced to the equation in noncritical case, and the solution of the class F of this equation is sought by the method of successive approximations using the principle compression reflections. Then based on the properties of the solutions of the transformed equation, conclusions are drawn about the properties of the initial equation.

1 Basic notation and definitions

Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in(\mathbf{0},\varepsilon_{\mathbf{0}}),\ \varepsilon_{\mathbf{0}}\in\mathbf{R}^+\big\}.$$

Definition 1.1. We say that a function $p(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$, $m\in\mathbf{N}\cup\{\mathbf{0}\}$, if:

(1) $p:G(\varepsilon_0)\to\mathbf{C}$;

(2) $p(t,\varepsilon)\in C^m(G(\varepsilon_0))$ at t;

(3) $\dfrac{d^kp(t,\varepsilon)}{dt^k}=\varepsilon^kp_k(t,\varepsilon)$ $(0\le k\le m)$,

$$\|p\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|p_k(t,\varepsilon)|<+\infty.$$

Definition 1.2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$, if

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\,\theta(t,\varepsilon)),$$

and

(1) $f_n(t,\varepsilon)\in S(m,\varepsilon_0)$ $(n\in\mathbf{Z})$;

(2)
$$\|f\|_{F(m;\varepsilon_0;\theta)} \overset{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty;$$

(3)
$$\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)\,d\tau, \quad \varphi \in \mathbf{R}^+, \ \ \varphi \in (m,\varepsilon_0), \ \ \inf_{G(\varepsilon_0)} \varphi(t,\varepsilon) = \varphi_0 > 0.$$

Definition 1.3. We say that a matrix $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ ($m \in \mathbf{N} \cup \{0\}$), if $a_{jk} \in S(m;\varepsilon_0)$ ($j,k = \overline{1,N}$).

We define the norm
$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)} = \max_{1\le j\le N} \sum_{k=1}^N \|a_{jk}(t,\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 1.4. We say that a matrix $B(t,\varepsilon,\theta) = (b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ ($m \in \mathbf{N} \cup \{0\}$), if $b_{jk}(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$ ($j,k = \overline{1,N}$).

We define the norm
$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N} \sum_{k=1}^N \|b_{jk}(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

2 Statement of the problem

Consider the matrix differential equation:
$$\frac{dX}{dt} = A(t,\varepsilon)X - XB(t,\varepsilon) + P(t,\varepsilon,\theta) + \mu\Phi(t,\varepsilon,\theta,X), \tag{2.1}$$
where X is an unknown square matrix of order N, belonging to some closed bounded domain $D \subset \mathbf{C}^{N\times N}$, where $\mathbf{C}^{N\times N}$ is the space of complex-valued matrices of dimension $N\times N$, $A(t,\varepsilon)$, $B(t,\varepsilon)$ belongs to the class $S_2(m;\varepsilon_0)$, $P(t,\varepsilon,\theta)$ belongs to the class $F_2(m;\varepsilon_0;\theta)$. $\Phi(t,\varepsilon,\theta,X)$ is a matrix function belonging to the class $_2(m;\varepsilon_0;\theta)$ with respect t,ε,θ and continuos with respect X in D. μ are real parameter.

We denote $\lambda_j^1(t,\varepsilon)$, $\lambda_j^2(t,\varepsilon)$ ($j = \overline{1,N}$) – eigenvalues, respectively, of matrices $A(t,\varepsilon)$, $B(t,\varepsilon)$, for which the following conditions are satisfied:

1^0.
$$\inf_{G(\varepsilon_0)} \big|\lambda_j^1(t,\varepsilon) - \lambda_k^1(t,\varepsilon) - in\,\varphi(t,\varepsilon)\big| \ge b_0 > 0,$$
$$\inf_{G(\varepsilon_0)} \big|\lambda_j^2(t,\varepsilon) - \lambda_k^2(t,\varepsilon) - in\,\varphi(t,\varepsilon)\big| \ge b_0 > 0 \ \ \forall\, n \in \mathbf{Z}, \ \ j,k = \overline{1,N}, \ \ j \ne k.$$

2^0.
$$\lambda_j^1(t,\varepsilon) - \lambda_k^2(t,\varepsilon) = i\omega_{jk}(t,\varepsilon), \ \ \omega_{jk}(t,\varepsilon) \in \mathbf{R},$$
$$\inf_{G(\varepsilon_0)} \big|\omega_{jk}(t,\varepsilon) - n\varphi(t,\varepsilon)\big| \ge b_0 > 0 \ \ \forall\, n \in \mathbf{Z}, \ \ j,k = \overline{1,N}.$$

We study the problem on the existence of particular solutions of classes $F_2(m_1;\varepsilon_1;\theta)$, $m_1 \le m$, $\varepsilon_1 \le \varepsilon_0$ of equation (2.1). The condition 2^0 shows that in this case we are dealing with a critical case.

3 Auxiliary results

Lemma 3.1. *Let*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + u(t,\varepsilon,\theta(t,\varepsilon)) \tag{3.1}$$

be a given scalar linear non-homogeneous first-order differential equation, where $\lambda(t,\varepsilon) \in S(m;\varepsilon)$, $\inf\limits_{G(\varepsilon_0)} |\operatorname{Re}\lambda(t,\varepsilon)| = \gamma > 0$, *and* $u(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$. *Then equation* (3.1) *has a unique particular solution* $x(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$. *This solution is given by the formula:*

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \int\limits_T^t u(\tau,\varepsilon,\theta(\tau,\varepsilon)) \exp\bigg(\int\limits_\tau^t \lambda(s,\varepsilon)\,ds\bigg)\,d\tau,$$

where

$$T = \begin{cases} -\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon) \le -\gamma < 0, \\ +\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon) \ge \gamma > 0. \end{cases}$$

Moreover, there exists $K_0 \in (0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)} \le K_0 \|u(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Lemma 3.2. *Let equation* (2.1) *satisfy the next conditions:*
1) there exist matrices $L_1(t,\varepsilon), L_2(t,\varepsilon) \in S_2(m;\varepsilon_0)$ *such that*

(a) $|\det L_k(t,\varepsilon)| \ge a_0 > 0$ $(k = 1,2)$;

(b) $L_1^{-1}(t,\varepsilon)A(t,\varepsilon)L_1(t,\varepsilon) = D_1(t,\varepsilon) = (d^1_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$;

(c) $L_2(t,\varepsilon)B(t,\varepsilon)L_2^{-1}(t,\varepsilon) = D_2(t,\varepsilon) = (d^2_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

where D_1, D_2 *are lower triangular matrices, belonging to the class* $S_2(m;\varepsilon_0)$,

$$d^1_{jj}(t,\varepsilon) = \lambda^1_j(t,\varepsilon), \quad d^2_{kk}(t,\varepsilon) = \lambda^2_k(t,\varepsilon).$$

Then by using the transformation

$$X = L_1(t,\varepsilon)YL_2(t,\varepsilon)$$

equation (2.1) leads to the form:

$$\frac{dY}{dt} = D_1(t,\varepsilon)Y - YD_2(t,\varepsilon) - \varepsilon H_1(t,\varepsilon)Y - \varepsilon YH_2(t,\varepsilon) + F_1(t,\varepsilon,\theta) + \mu\Phi_1(t,\varepsilon,\theta,Y), \tag{3.2}$$

where

$$H_1(t,\varepsilon) = \frac{1}{\varepsilon}\,L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}\,, \quad H_2(t,\varepsilon) = \frac{1}{\varepsilon}\,\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon),$$
$$F_1(t,\varepsilon,\theta) = L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon),$$
$$\Phi_1(t,\varepsilon,\theta,Y) = L_1^{-1}(t,\varepsilon)\Phi(t,\varepsilon,\theta,L_1(t,\varepsilon)YL_2(t,\varepsilon))L_2^{-1}(t,\varepsilon).$$

Lemma 3.3. *Let a linear matrix equation be given*

$$\frac{dX}{dt} = \Big(D_1(t,\varepsilon) + \sum_{l=1}^q B_{1l}(t,\varepsilon,\theta)\mu^l\Big)X - X\Big(D_2(t,\varepsilon) + \sum_{l=1}^q B_{2l}(t,\varepsilon,\theta)\mu^l\Big), \tag{3.3}$$

$D_1(t,\varepsilon), D_2(t,\varepsilon)$ – the same as in Lemma 3.2, $B_{1l}(t,\varepsilon,\theta)$, $B_{2l}(t,\varepsilon,\theta)$ ($l=\overline{1,q}$) belong to the class $F_2(m;\varepsilon_0;\theta)$, $\mu\in(0,1)$ is a small real parameter. Then for sufficiently small values μ there exists transformation

$$X=\Big(E+\sum_{l=1}^{q}Q_{1l}(t,\varepsilon,\theta)\mu^l\Big)Y\Big(E+\sum_{l=1}^{q}Q_{2l}(t,\varepsilon,\theta)\mu^l\Big),$$

where $Q_{1l}(t,\varepsilon,\theta)$, $Q_{2l}(t,\varepsilon,\theta)$ ($l=\overline{1,q}$) belong to the class $F_2(m;\varepsilon_0;\theta)$, which leads equation (3.2) to the form

$$\begin{aligned}\frac{dY}{dt}=\Big(D_1(t,\varepsilon)+\sum_{l=1}^{q}U_{1l}(t,\varepsilon)\mu^l+\varepsilon\sum_{l=1}^{q}V_{1l}(t,\varepsilon,\theta)\mu^l+\mu^{q+1}W_1(t,\varepsilon,\theta,\mu)\Big)Y-\\ -Y\Big(D_2(t,\varepsilon)+\sum_{l=1}^{q}U_{2l}(t,\varepsilon)\mu^l+\varepsilon\sum_{l=1}^{q}V_{2l}(t,\varepsilon,\theta)\mu^l+\mu^{q+1}W_2(t,\varepsilon,\theta,\mu)\Big),\end{aligned}$$

where $U_{1l}(t,\varepsilon)$, $U_{2l}(t,\varepsilon)$ ($l=\overline{1,q}$) are diagonal matrices which belong to the class $S_2(m;\varepsilon_0)$, V_{1l}, V_{2l}, W_1, W_2 ($l=\overline{1,q}$) are square matrices which belong to the class $F(m-1;\varepsilon_0;\theta)$.

Lemma 3.4. *Let a matrix-function $\Phi_1(t,\varepsilon,\theta,Y)$ in equation (3.2) have in D^* continuous derivatives with respect to Y in the sense of Frechet up to order $2q+1$ inclusive, and if $Y\in F_2(m;\varepsilon_0;\theta)$, then these derivatives are also from the class $F_2(m;\varepsilon_0;\theta)$. Then there exists $\mu_0\in(0,1)$ such that for all $\mu_1\in(0,\mu_0)$ there exists the transformation*

$$Y=\Psi_1(t,\varepsilon,\theta,\mu)+\Psi_2(t,\varepsilon,\theta,\mu)Z\Psi_3(t,\varepsilon,\theta,\mu), \tag{3.4}$$

where $Z\in D^{**}\subset \mathbf{C}^{N\times N}$, $\Psi_1(t,\varepsilon,\theta,\mu),\Psi_2(t,\varepsilon,\theta,\mu),\Psi_3(t,\varepsilon,\theta,\mu)\in F_2(m;\varepsilon_0;\theta)$, which leads equation (3.2) to the form:

$$\begin{aligned}\frac{dZ}{dt}=\Big(D_1(t,\varepsilon)+\sum_{l=1}^{q}U_{1l}(t,\varepsilon)\mu^l\Big)Z-Z\Big(D_2(t,\varepsilon)+\sum_{l=1}^{q}U_{2l}(t,\varepsilon)\mu^l\Big)\\ +\varepsilon K(t,\varepsilon,\theta,\mu)+\mu^{2q}C(t,\varepsilon,\theta,\mu)+\varepsilon V_1(t,\varepsilon,\theta,\mu)Z-\varepsilon ZV_2(t,\varepsilon,\theta,\mu)\\ +\mu^{q+1}\big(R_1(t,\varepsilon,\theta,\mu)Z-ZR_2(t,\varepsilon,\theta,\mu)\big)+\mu\Phi_2(t,\varepsilon,\theta,Z,\mu),\end{aligned} \tag{3.5}$$

where $K\in F_2(m-1;\varepsilon_0;\theta)$, $U_{1l},U_{2l}\in S_2(m;\varepsilon_0)$, $R_1,R_2,C\in F_2(m;\varepsilon_0;\theta)$, $V_1,V_2\in F(m-1;\varepsilon_0;\theta)$, matrix-function Φ_2 belong to class $F_2(m;\varepsilon_0;\theta)$ with respect t, ε, θ, continuously differentiable in the sense of Frechet with respect Z and contains terms of at least second order with respect to Z.

4 Basic results

Theorem 4.1. *Let equation (3.5) be such that there exists $q_0\in\mathbf{N}$ ($1\le q_0\le N$) such that*

$$\inf_{G(\varepsilon_0)}\Big|\operatorname{Re}\big((U_{1q_0}(t,\varepsilon))_{jj}-(U_{2q_0}(t,\varepsilon))_{kk}\big)\Big|\ge b_0>0\ \ (j,k=\overline{1,N}),$$

and for all $l=\overline{1,q_0-1}$ (if $q_0>1$):

$$\operatorname{Re}\big((U_{1l}(t,\varepsilon))_{jj}-(U_{2l}(t,\varepsilon))_{kk}\big)\equiv 0\ \ (j,k=\overline{1,N}).$$

Then there exists $\mu_3\in(0,1)$, $\varepsilon_1(\mu)\in(0,\varepsilon_0)$ such that for all $\mu\in(0,\mu_3)$, $\varepsilon\in(0,\varepsilon_1(\mu))$ there exist a particular solution of equation (3.5) which belongs to the class $F_2(m-1;\varepsilon_1(\mu);\theta)$.

Theorem 4.2. *Let equation* (2.1) *be such that the following conditions are met:*

(1) *conditions* 1^0, 2^0*;*

(2) *conditions of Lemma 3.2;*

(3) *equation* (3.2) *satisfies the conditions of Lemma 3.4;*

(4) *equation* (3.5) *satisfies the conditions of Theorem 4.1.*

Then there exist $\mu_4 \in (0,1)$*,* $\varepsilon_4(\mu) \in (0,\varepsilon_0)$ *such that for all* $\mu \in (0,\mu_4)$ *and for all* $\varepsilon \in (0,\varepsilon_4(\mu))$ *there exist a particular solution of equation* (2.1) *which belongs to the class* $F_2(m-1;\varepsilon_4(\mu);\theta)$*.*