On the Representation of a Solution for the Perturbed Controlled Differential Equation with the Discontinuous Initial Condition Considering Perturbation of the Initial Moment

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In the paper, for the perturbed controlled nonlinear differential equation with the constant delay in the phase coordinates and in controls a formula of the analytic representation of a solution is obtained in the left semi-neighborhood of the endpoint of the main interval. The novelty here is the effect in the formula related with perturbation of the initial moment. Analogous formulas without perturbation of the initial moment and without delay in controls are given in [1, 3, 4].

Let $I=[0,T]$ be a finite interval and let $\tau_2>\tau_1>0$, and $\theta>0$ be given numbers; suppose that $O\subset\mathbb{R}^n$ and $U\subset\mathbb{R}^r$ are open sets. Let n-dimensional function $f(t,x,y,u,v)$ be continuous on $I\times O^2\times U^2$ and continuously differentiable with respect to x, y, u and v. Let Φ be a set of continuously differentiable initial functions $\varphi: I_1=[-\tau_2,T]\to O$ and let Ω be a set of piecewise-continuous and bounded control functions $u(t)\in U$, $t\in I_2=[-\theta,T]$.

In the space $\mathbb{R}^n$ to each element $\mu=(t_0,\tau,x_0,\varphi,u)\in\Lambda=[0,T)\times[\tau_1,\tau_2]\times O\times\Phi\times\Omega$ we assign the delay controlled differential equation

$$\dot{x}(t)=f(t,x(t),x(t-\tau),u(t),u(t-\theta)),\ \ t\in[t_0,T] \tag{1}$$

with the discontinuous initial condition

$$x(t)=\varphi(t),\ \ t\in[-\tau_2,t_0),\quad x(t_0)=x_0. \tag{2}$$

Condition (2) is called the discontinuous initial condition because, in general, $x(t_0)\neq\varphi(x_0)$.

Definition. Let $\mu=(t_0,\tau,x_0,\varphi,u)\in\Lambda$. A function $x(t)=x(t;\mu)\in O$, $t\in I_1$ is called a solution of equation (1) with the initial condition (2) or a solution corresponding to μ and defined on the interval I_1 if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,T]$ and satisfies equation (1) almost everywhere on $[t_0,T]$.

It is clear that the solution $x(t)=x(t;\mu)$, $t\in I_1$, in general, at the point t_0 is discontinuous. Let us introduce notations

$$|\mu|=|t_0|+|\tau|+|x_0|+\|\varphi\|_1+\|u\|,\quad \Lambda_\varepsilon(\mu_0)=\big\{\mu\in\Lambda:\ |\mu-\mu_0|\le\varepsilon\big\},$$

where
$$\|\varphi\|_1 = \sup\big\{|\varphi(t)| + |\dot\varphi(t)| :\ t \in I_1\big\}, \quad \|u\| = \sup\big\{|u(t)| :\ t \in I_2\big\},$$
$\varepsilon > 0$ is a fixed number and $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, u_0) \in \Lambda$ is a fixed element; furthermore,
$$\delta t_0 = t_0 - t_{00},\ \ \delta x_0 = x_0 - x_{00},\ \ \delta\varphi(t) = \varphi(t) - \varphi_0(t),\ \ \delta u(t) = u(t) - u_0(t),$$
$$\delta\mu = \mu - \mu_0 = (\delta t_0, \delta\tau, \delta x_0, \delta\varphi, \delta u), \quad |\delta\mu| = |\delta t_0| + |\delta\tau| + |\delta x_0| + \|\delta\varphi\|_1 + \|\delta u\|.$$

Remark. Let $x(t;\mu_0)$ be the solution corresponding to $\mu_0 \in \Lambda$ and defined on the interval I_1, i.e. $x(t;\mu_0)$ is the solution of the problem
$$\dot x(t) = f\big(t, x(t), x(t-\tau_0), u_0(t), u_0(t-\theta)\big),\ \ t \in [t_{00}, T],$$
$$x(t) = \varphi_0(t),\ \ t \in [-\tau_2, t_{00}), \quad x(t_{00}) = x_{00}.$$

Then, there exists a number $\varepsilon_1 > 0$ such that to each element $\mu \in \Lambda_{\varepsilon_1}(\mu_0)$ there corresponds the solution $x(t;\mu)$ defined on the interval I_1, i.e. the perturbed problem (1), (2) has the solution $x(t;\mu)$, $t \in I_1$ [2, p. 18].

Theorem 1. *Let $x_0(t) := x(t;\mu_0)$ be the solution corresponding to the element $\mu_0 = (t_{00}, \tau_0, x_{00}, \varphi_0, u_0) \in \Lambda$ and defined on the interval I_1, with $t_{00} + \tau_0 < T$. Let $\delta > 0$ and $\varepsilon_2 \in (0, \varepsilon_1)$ be numbers such that $t_{00} + \tau_0 + \varepsilon_2 < T - \delta$. Then, for arbitrary*
$$\mu \in \Lambda^-_{\varepsilon_2}(\mu_0) = \big\{\mu = (t_0, \tau, x_0, \varphi, u) \in \Lambda_{\varepsilon_2}(\mu_0) :\ 0 \le t_0 \le t_{00}\big\}$$
on the interval $[T-\delta, T]$, the following representation holds
$$x(t;\mu) = x_0(t) + \delta x^-(t;\delta\mu) + o(t;\delta\mu), \tag{3}$$
where
$$\begin{aligned}
\delta x^-(t;\delta\mu) &= -\big[Y(t_{00};t)f_0^- + Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0 - Y(t_{00}+\tau_0;t)f_1^-\delta\tau + \beta(t;\delta\mu),\\
\beta(t;\delta\mu) &= Y(t_{00};t)\delta x_0\\
&\quad + \int_{t_{00}-\tau_0}^{t_{00}} Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)\,ds + \bigg[\int_{t_{00}}^{t} Y(s;t)f_y[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau\\
&\quad + \int_{t_{00}}^{t} Y(s;t)f_u[s]\delta u(s)\,ds + \int_{t_{00}}^{t} Y(s;t)f_v[s]\delta u(s-\theta)\,ds,\\
f_0^- &= f\big(t_{00}, x_{00}, \varphi_0(t_{00}-\tau_0), u_0(t_{00}-), u_0(t_{00}-\theta-)\big),\\
f_1^- &= f\big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), x_{00}, u_0(t_{00}+\tau_0-), u_0(t_{00}+\tau_0-\theta-)\big)\\
&\qquad - f\big(t_{00}+\tau_0, x_0(t_{00}+\tau_0), \varphi_0(t_{00}), u_0(t_{00}+\tau_0-), u_0(t_{00}+\tau_0-\theta-)\big),\\
f_y[s] &= f_y\big(s, x_0(s), x_0(s-\tau_0), u_0(s), u_0(t_{00}+\tau_0-\theta-)\big),\\
&\lim_{|\delta\mu|\to 0} \frac{|o(t;\delta\mu)|}{|\delta\mu|} = 0 \ \ \textit{uniformly for}\ \ t \in [T-\delta, T].
\end{aligned} \tag{4}$$
Next, $Y(s;t)$ is the $n\times n$-matrix function satisfying the equation
$$Y_s(s;t) = -Y(s;t)f_x[s] - Y(s+\tau_0;t)f_y[s+\tau_0],\ \ s \in [t_{00}, t]$$
and the condition
$$Y(s;t) = \begin{cases} H & \textit{for}\ \ s = t,\\ \Theta & \textit{fors} > t; \end{cases}$$
H is the identity matrix and Θ is the zero matrix.

Some comments

The function $\delta x^-(t;\delta\mu)$ is called the first variation of a solution $x_0(t)$, $t\in[t_{10}-\delta_2,t_{10}+\delta_2]$. The expression (4) is called the variation formula of a solution. The addend

$$-\big[Y(t_{00};t)f_0^-+Y(t_{00}+\tau_0;t)f_1^-\big]\delta t_0$$

in (4) *is the effect of perturbation of the initial moment* t_{00}. Namely, here f_1^- is the effect of the discontinuous initial condition (2). The addend

$$-\bigg[Y(t_{00}+\tau_0;t)f_1^-+\int\limits_{t_{00}}^{t}Y(s;t)f_y[s]\dot x_0(s-\tau_0)\,ds\bigg]\delta\tau$$

in (4) is the effect of perturbation of the delay τ_0. The expression

$$Y(t_{00};t)\delta x_0+\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(s+\tau_0;t)f_y[s+\tau_0]\delta\varphi(s)\,ds$$

in (4) is the effect of perturbations of the initial vector x_{00} and the initial function $\varphi_0(t)$. The expression

$$\int\limits_{t_{00}}^{t}Y(s;t)\big[f_u[s]\delta u(s)+f_v[s]\delta u(s-\theta)\big]\,ds$$

in (4) is the effect of perturbation of the control function $u_0(t)$.

Formula (3) allows us to obtain an approximate solution of the perturbed problem (1), (2) in the analytical form on the interval $T-\delta,T]$. In fact, for a small $|\delta\mu|$ from (3) it follows

$$x(t;\mu)\approx x_0(t)+\delta x^-(t;\delta\mu),\ \ t\in[T-\delta,T].$$

Theorem 2. *Let $x_0(t):=x(t;\mu_0)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_{00}+\tau_0<T$. Let $\delta>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ be numbers such that $t_{00}+\tau_0+\varepsilon_2<T-\delta$. Then, for arbitrary*

$$\mu\in\Lambda^+_{\varepsilon_2}(\mu_0)=\big\{\mu=(t_0,\tau,x_0,\varphi,u)\in\Lambda_{\varepsilon_2}(\mu_0):\ t_{00}\le t_0<T\big\}$$

on the interval $[T-\delta,T]$, the following representation holds

$$x(t;\mu)=x_0(t)+\delta x^+(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\delta x^+(t;\delta\mu)=-\big[Y(t_{00};t)f_0^++Y(t_{00}+\tau_0;t)f_1^+\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1^+\delta\tau+\beta(t;\delta\mu).$$

Theorem 3. *Let $x_0(t):=x(t;\mu_0)$ be the solution corresponding to the element $\mu_0=(t_{00},\tau_0,x_{00},\varphi_0,u_0)\in\Lambda$ and defined on the interval I_1, with $t_{00}+\tau_0<T$. Let the functions $u_0(t)$ and $u_0(t-\theta)$ are continuous at the points t_{00} and $t_{00}+\tau_0$. Besides, let $\delta>0$ and $\varepsilon_2\in(0,\varepsilon_1)$ be numbers such that $t_{00}+\tau_0+\varepsilon_2<T-\delta$. Then, for arbitrary $\mu\in\Lambda_{\varepsilon_2}(\mu_0)$ on the interval $[T-\delta,T]$, the following representation holds*

$$x(t;\mu)=x_0(t)+\delta x(t;\delta\mu)+o(t;\delta\mu),$$

where

$$\begin{aligned}\delta x(t;\delta\mu)&=-\big[Y(t_{00};t)f_0+Y(t_{00}+\tau_0;t)f_1\big]\delta t_0-Y(t_{00}+\tau_0;t)f_1\delta\tau+\beta(t;\delta\mu),\\ f_0&=f\big(t_{00},x_{00},\varphi_0(t_{00}-\tau_0),u_0(t_{00}),u_0(t_{00}-\theta)\big),\\ f_1&=f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),x_{00},u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta)\big)\\ &\qquad-f\big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta)\big).\end{aligned}$$

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