Investigation of the Oscillation, Rotation and Wandering Radial Indicators of a Differential System by the First Approximation

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For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider a nonlinear, generally speaking, differential system of the form

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \;\; t \in \mathbb{R}_+ \equiv [0,+\infty), \;\; x \in G, \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed. We associate with system (1) the linear homogeneous system of its *first approximation*

$$\dot{x} = A(t)x, \quad A(t) \equiv f'_x(t,0), \;\; t \in \mathbb{R}_+, \;\; x \in \mathbb{R}^n, \tag{2}$$

for which we do not require here the uniformity in $t \in \mathbb{R}_+$ of the natural (pointwise) smallness of the nonlinear addition

$$h(t,x) \equiv f(t,x) - A(t)x = o(x), \;\; x \to 0.$$

Let $x_f(\,\cdot\,, x_0)$ be a non-extendable solution of system (1) with the initial condition $x_f(0,x_0) = x_0$. By $S_*(f)$ and S_A we denote the set of all nonzero solutions to system (1) and, accordingly, the set of all solutions to system (2).

Definition 1. Let us list three basic [1] *functional* $\mathrm{K}(t,u)$ defined on the pairs $t \in \mathbb{R}_+$ and $u : [0,t] \to \mathbb{R}^n$ (taking the value $+\infty$ whenever the function is not defined on the entire segment $[0,t]$), corresponding to *indicators*

$$\varkappa = \nu, \theta, \rho, \;\; \text{respectively, for} \;\; \mathrm{K} = \mathrm{N}, \Theta, \mathrm{P}, \tag{3}$$

and describing the following properties of solutions:

1) *oscillation* ($\varkappa = \nu$), if $\mathrm{K}(t,u) = \mathrm{N}(t,u)$ is the number (multiplied by π) zeros of the function $P_1 u$ on the interval $(0,t]$, where P_1 is an orthogonal projector onto a fixed line, and if at least one of these zeros is *multiple* (that is, it is also a zero and derivative $(P_1 u)^{\cdot}$), then we assume $\mathrm{N}(t,u) = +\infty$;

2) *rotation* (*oriented*, $\varkappa = \theta$), if $\mathrm{K}(t,u) = \Theta(t,u) \equiv |\varphi(t, P_2 u)|$ is module of *oriented* angle $\varphi(t, P_2 u)$ (continuous in t, with initial condition $\varphi(0, P_2 u) = 0$) between the vector $P_2 u(t)$ and the initial vector $P_2 u(0)$, where P_2 is the orthogonal projector onto a fixed two-dimensional plane, and if $P_2 u(\tau) = 0$ for at least one $\tau \in [0,t]$, then we assume $\Theta(t,u) = +\infty$;

3) *wandering* ($\varkappa = \rho$), if

$$\mathrm{K}(t,u) = \mathrm{P}(t,u) \equiv \int_0^t \big| \big(u(\tau)/|u(\tau)|\big)^{\cdot} \big| \, d\tau, \;\; u(\tau) \neq 0, \;\; \tau \in [0,t].$$

There are also known the other functionals that are responsible for the *non-oriented* or *frequency rotation* [1], *k-th rank rotation* [2], and *plane rotation* [3].

Definition 2 ([4])**.** For each functional described in Definition 1, we define:

(a) *weak* and *strong lower linear* indicators (3) of the solution $x \in S_*(f)$ defined on the whole semiaxis $\mathbb{R}_+$ – by the formulas

$$\check{\varkappa}^\circ(x) \equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\mathrm{K}(t,Lx), \quad \check{\varkappa}^\bullet(x) \equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\mathrm{K}(t,Lx); \tag{4}$$

(b) *weak* and *strong lower radial* indicators (3) of the Cauchy problem for system (1) with the initial value $x_0 \in G$ – by the formulas

$$\begin{aligned} \check{\varkappa}_r^\circ(f,x_0) &\equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\check{\mathrm{K}}_r(f,x_0,t,L), \\ \check{\varkappa}_r^\bullet(f,x_0) &\equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\check{\mathrm{K}}_r(f,x_0,t,L), \end{aligned} \tag{5}$$

where

$$\check{\mathrm{K}}_r(f,x_0,t,L) = \varliminf_{\mu\to+0} \mathrm{K}(t,Lx_f(\,\cdot\,,\mu x_0)); \tag{6}$$

(c) *weak* and *strong lower spherical* indicators (3) of the Cauchy problem for system (1) with the initial value $x_0 \in G$ – by the formulas

$$\begin{aligned} \check{\varkappa}_s^\circ(f,x_0) &\equiv \varliminf_{t\to+\infty} \inf_{L\in\operatorname{Aut}\mathbb{R}^n} t^{-1}\check{\mathrm{K}}_s(f,x_0,t,L), \\ \check{\varkappa}_s^\bullet(f,x_0) &\equiv \inf_{L\in\operatorname{Aut}\mathbb{R}^n} \varliminf_{t\to+\infty} t^{-1}\check{\mathrm{K}}_s(f,x_0,t,L), \end{aligned} \tag{7}$$

where

$$\mathrm{K}_s(f,x_0,t,L) \equiv \mathrm{K}(t,Lx_{f_s}(\,\cdot\,,x_0)), \quad f_s(t,x) \equiv P_x^\perp f(t,x),$$

$P_x^\perp$ is a projector onto a hyperplane orthogonal to x, and the modified system

$$\dot{x} = f_s(t,x), \quad (t,x) \in \mathbb{R}_+ \times G, \tag{8}$$

is also called *spherical* (with respect to system (1));

(d) *weak* and *strong upper* indicators – *linear* $\widehat{\varkappa}^\circ(x)$, $\widehat{\varkappa}^\bullet(x)$, *radial* $\widehat{\varkappa}_r^\circ(f,x_0)$, $\widehat{\varkappa}_r^\bullet(f,x_0)$ and *spherical* $\widehat{\varkappa}_s^\circ(f,x_0)$, $\widehat{\varkappa}_s^\bullet(f,x_0)$ – by the same formulas (3), (5) and (7), respectively, but with the replacement in formulas (3)–(7) of all lower limits for $t\to+\infty$ and for $\mu\to+0$ by upper ones;

(e) *exact* or *absolute* varieties of the same indicators that arise when the corresponding values of the lower and upper indicators or, respectively, weak and strong ones coincide: in the first case, we will omit the checkmark and the cap in their designation, and in the second one – an empty and full circle.

Everywhere below, the letters $\varkappa$ or K mean any (corresponding) of the indicators or functionals (3), and the top icons ~ or $*$ are any of the icons $\check{}$, $\hat{}$ or $\circ$, $\bullet$, respectively.

The introduction of radial and spherical indicators (as well as *ball* ones [4]) is due to the fact that some solutions of the nonlinear system (1) may be defined not on the entire time semiaxis.

On the one hand, for linear systems, the linear and nonlinear (radial and spherical) indicators are indistinguishable.

Theorem 1. *If system* (1) *is linear homogeneous and* $G = \mathbb{R}^n$, *then for any solution* $x \in S_*(f)$ *the equalities hold*

$$\widetilde{\mathrm{K}}_r(f, x(0), t, L) = \mathrm{K}_s(f, x(0), t, L) = \mathrm{K}(f, Lx), \quad t \in \mathbb{R}_+, \quad \operatorname{Aut}\mathbb{R}^n,$$
$$\widetilde{\varkappa}_r^*(f, x(0)) = \widetilde{\varkappa}_s^*(f, x(0)) = \widetilde{\varkappa}^*(x).$$

On the other hand, in the nonlinear (even if autonomous) case, that coincidence is no longer observed.

Theorem 2. *If* $n = 2$ *and* $G = \mathbb{R}^2$, *then for each of the following four lines of relations separately*

$$\begin{aligned} 0 = \varkappa_r(f, x(0)) < \varkappa_s(f, x(0)) < \varkappa(x) = +\infty, \\ 0 = \varkappa_r(f, x(0)) = \varkappa(x) < \varkappa_s(f, x(0)) < +\infty, \\ 1 = \varkappa_r(f, x(0)) > \varkappa_s(f, x(0)) > \varkappa(x) = 0, \\ 1 = \varkappa_r(f, x(0)) = \varkappa(x) > \varkappa_s(f, x(0)) > 0, \end{aligned}$$

there exists an autonomous system (1) *such that any solution* $x \in S_*(f)$ *is defined on* $\mathbb{R}_+$, *and all linear, radial and spherical indicators are exact, absolute and satisfy the relations of that particular line.*

The radial wandering indicators completely coincide with the corresponding linear ones of the first approximation system.

Theorem 3. *For any system* (1) *and any nonzero solution* $x \in S_A$ *to the system of its first approximation* (2), *the equalities hold*

$$\check{\mathrm{P}}_r(f, x(0), t, L) = \widehat{\mathrm{P}}_r(f, x(0), t, L) = \mathrm{P}(Lx, t), \quad t \in \mathbb{R}_+, \quad L \in \operatorname{Aut}\mathbb{R}^n,$$
$$\widetilde{\rho}_r^*(f, x(0)) = \widetilde{\rho}^*(x).$$

In the two-dimensional case, a similar coincidence is observed also for the rotation indicators.

Theorem 4. *If* $n = 2$, *then for any system* (1) *and any nonzero solution* $x \in S_A$ *to the system of its first approximation* (2), *the equalities hold*

$$\check{\Theta}_r(f, x(0), t, L) = \widehat{\Theta}_r(f, x(0), t, L) = \Theta(Lx, t), \quad t \in \mathbb{R}_+, \quad L \in \operatorname{Aut}\mathbb{R}^n,$$
$$\widetilde{\theta}_r^*(f, x(0)) = \widetilde{\theta}^*(x).$$

However, already in the three-dimensional (and even autonomous) case, the rotational radial indicators, as well as the oscillation ones, generally speaking, do not match the linear ones.

Theorem 5. *For* $n = 3$ *and* $G = \mathbb{R}^3$ *there exists an autonomous system* (1) *such that for any nonzero solution* $x \in S_A$ *of the system of its first approximation* (2) *the solution* $x_f(\,\cdot\,, x(0))$ *is also defined on* $\mathbb{R}_+$, *and all the rotational and oscillation indicators are exact, absolute and for some two-dimensional subspace* $S \subset S_A$ *satisfy the relations*

$$0 = \theta_r(f, x(0)) = \nu_r(f, x(0)) \leqslant \theta(x) = \nu(x) = \begin{cases} 1, & x \in S \setminus \{0\}; \\ 0, & x \notin S. \end{cases}$$

For the linear and nonlinear radial indicators of oscillation, a similar mismatch is observed already in the two-dimensional (albeit only in a non-autonomous) case.

Theorem 6. *For $n = 2$ and $G = \mathbb{R}^2$ there exists a system* (1) *such that for any solution $x \in S_A$ of the system of its first approximation* (2) *the solution $x_f(\,\cdot\,, x(0))$ is also defined on $\mathbb{R}_+$, and all the linear and radial oscillation indicators are exact, absolute and satisfy the relations*

$$0 = \nu_r(f, x(0)) < \nu(x) = 1.$$

For the spherical indicators, however, no analogs of Theorems 3 and 4 above are valid (that follows from Theorems 2 and 3).

Theorem 7. *If $n = 2$ and $G = \mathbb{R}^2$, then for each of the following two lines of relations separately*

$$0 = \varkappa(x) < \varkappa_s(f, x(0)) < +\infty,$$
$$1 = \varkappa(x) > \varkappa_s(f, x(0)) > 0,$$

there exists an autonomous system (1) *such that for any nonzero solution $x \in S_A$ of the system of its first approximation* (2) *the solution $x_f(\,\cdot\,, x(0))$ is also defined on $\mathbb{R}_+$, and all the linear and spherical indicators are exact, absolute and satisfy the relations of that particular line.*

References

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