Description of the Linear Perron Effect Under Parametric Perturbations of a System with Unbounded Coefficients

A. Ravcheev

Lomonosov Moscow State University, Moscow, Russia
E-mail: rav4eev@yandex.ru

For a given $n \in \mathbb{N}$ let us denote by $\widetilde{\mathcal{M}}_n$ the set of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty). \tag{1}$$

with continuous matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, which we identify with the corresponding linear systems. The subset of $\widetilde{\mathcal{M}}_n$ that consists of systems (1) with bounded coefficients will be denoted by $\mathcal{M}_n$.

In [8] O. Perron constructed a two-dimensional system $A \in \mathcal{M}_2$ with negative Lyapunov exponents and an exponentially decaying at infinity continuous matrix-valued function Q such that the Lyapunov exponents of the perturbed system $A + Q$ are greater than those of the original system A. Perron's studies (see also [9]) have become a starting point of deeper researches on dependency of the Lyapunov exponents on perturbations of different classes.

The phenomenon of abrupt change of the Lyapunov exponents of a system in $\mathcal{M}_n$ under a small perturbation was called in the monograph [6, Ch. 4] the *Perron effect*. Since the paper [5], this term is being used only for the case when perturbations do not decrease the Lyapunov exponents of the original system. Unlike [5, 6, 9], which consider higher-order perturbations, we study the Perron effect under linear perturbations and hence call it *linear* [2].

Let us recall that *the characteristic exponent* [1, p. 25] of a vector-function $f : P \to \mathbb{R}^n$, where P is an unbounded subset of the semi-axis $\mathbb{R}_+$, is the quantity (we assume that $\ln 0 = -\infty$)

$$\lambda[f] = \varlimsup_{P \ni t \to +\infty} \ln \|f(t)\|^{1/t},$$

and the *Lyapunov exponents* [7] of a system $A \in \widetilde{\mathcal{M}}_n$ are the quantities

$$\lambda_i(A) = \inf_{L \in G_i(S(A))} \sup_{x \in L} \lambda[x], \ \ i = 1, \dots, n,$$

$S(A)$ being the space of solutions of system (1) and $G_i(S(A))$ the set of i-dimensional subspaces of $S(A)$.

The spectrum of the Lyapunov exponents of system (1) is the n-tuple $\Lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$. As coefficients of systems under consideration are not supposed to be bounded, the Lyapunov exponents of these systems are points of the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$ with the standard order and topology.

As a more general case, for an arbitrary metric space M, let us consider a parametric family of linear differential systems

$$\dot{x} = A(t, \mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{2}$$

depending on a parameter $\mu \in M$ such that for each fixed $\mu \in M$ system (2) has continuous coefficients. Fixing $i = 1, \dots, n$ and assigning to each $\mu \in M$ the i-th Lyapunov exponent of system (2) we obtain the function $\lambda_i(A,\,\cdot\,): M \to \overline{\mathbb{R}}$ which is called *the i-th Lyapunov exponent of family* (2). Accordingly, the function $\Lambda(A,\,\cdot\,) = (\lambda_1(A,\,\cdot\,),\dots,\lambda_n(A,\,\cdot\,))$ is called *the spectrum of the Lyapunov exponents of family* (2).

Henceforth we will consider parametric families of linear differential systems of the form

$$\dot{x} = \big(A(t) + Q(t,\mu)\big)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty),$$

where $Q(\,\cdot\,,\,\cdot\,): \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is called a *parametric perturbation* of system (1).

As previously, let M be a metric space. For a given $\theta : \mathbb{R}_+ \to \mathbb{R}$ we denote by $\mathcal{Q}_n^\theta(M)$ the class of jointly continuous functions $Q(\,\cdot\,,\,\cdot\,): \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ such that each function Q for some $C_Q > 0$ satisfies the condition

$$\sup_{\mu\in M} \|Q(t,\mu)\| \leqslant C_Q e^{-\theta(t)t}, \ \ t \in \mathbb{R}_+.$$

For each $A \in \widetilde{\mathcal{M}}_n$, let

$$\mathcal{Q}_n^\theta[A](M) = \Big\{ Q \in \mathcal{Q}_n^\theta(M) \mid \ \ \forall\, i = 1,\dots,n, \ \forall\, \mu \in M \ \ \lambda_i(A+Q,\mu) \geqslant \lambda_i(A) \Big\}. \tag{3}$$

Put simply, the set $\mathcal{Q}_n^\theta[A](M)$ is the subset of $\mathcal{Q}_n^\theta(M)$ consisting of those perturbations that don't decrease the Lyapunov exponents of the original system A. Note that for each system $A \in \mathcal{M}_n$ the class $\mathcal{Q}_n^\theta[A](M)$ is nonempty since it contains the matrix $Q \equiv 0$.

It is of interest to describe in terms of the descriptive set theory the set of pairs composed of the spectrum of the Lyapunov exponents of a system A and that of a family of perturbed systems $A+Q$, where $A \in \widetilde{\mathcal{M}}_n$ and $Q \in \mathcal{Q}_n^\theta[A](M)$, i.e. the set

$$\Pi\mathcal{Q}_n^\theta(M) = \Big\{ (\Lambda(A), \Lambda(A+Q,\,\cdot\,)) \mid \ \ A \in \widetilde{\mathcal{M}}_n, \ Q \in \mathcal{Q}_n^\theta[A](M) \Big\}. \tag{4}$$

Let us recall some necessary set theory notation. We say [3, p. 267] that the function $f : M \to \overline{\mathbb{R}}$ belongs to the class $(^*, G_\delta)$ if for each $r \in \mathbb{R}$, the inverse image $f^{-1}\big([r,+\infty]\big)$ of the semi-interval $[r,+\infty]$ is a G_δ-set in M (i.e. it can be represented as countable intersection of open sets). In particular, the class $(^*, G_\delta)$ is a subclass of the second Baire class [3, p. 294].

The sought description of set (4) is contained in the following

Theorem 1. *For any metric space M, number $n \geqslant 2$ and continuous function $\theta : \mathbb{R}_+ \to \mathbb{R}$ the pair $\big(l, f(\,\cdot\,)\big)$, where $l = (l_1,\dots,l_n) \in (\overline{\mathbb{R}})^n$ and $f(\,\cdot\,) = (f_1(\,\cdot\,),\dots,f_n(\,\cdot\,)) : M \to (\overline{\mathbb{R}})^n$, belongs to the set $\Pi\mathcal{Q}_n^\theta(M)$ if and only if the following conditions are satisfied:*

1) *$l_1 \leqslant \cdots \leqslant l_n$;*

2) *$f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ for each $\mu \in M$;*

3) *$f_i(\mu) \geqslant l_i$ for all $\mu \in M$ and $i = 1,\dots,n$;*

4) *for each $i = 1,\dots,n$ the function $f_i(\,\cdot\,) : M \to \overline{\mathbb{R}}$ belongs to the class $(^*, G_\delta)$.*

Note that a similar result for systems with bounded coefficients is obtained in [2].

As an important application of the stated theorem, consider the following problem. Let Φ be the set of all continuous functions $\varphi : \mathbb{R}_+ \to (0,+\infty)$. For an arbitrary metric space M and a subset $\Psi \subset \Phi$ let $\mathcal{Q}_n[\Psi](M)$ denote the class consisting of continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the condition

$$\lim_{t\to+\infty} (\psi(t))^{-1} \sup_{\mu\in M} \|Q(t,\mu)\| = 0 \ \text{ for each } \ \psi \in \Psi.$$

Next, for each $A \in \widetilde{\mathcal{M}}_n$, let $\mathcal{Q}_n[\Psi, A](M)$ denote the subset of $\mathcal{Q}_n[\Psi](M)$ that consists of parametric perturbations that don't decrease the Lyapunov exponents of the system A, i.e.

$$\mathcal{Q}_n[\Psi, A](M) = \Big\{ Q \in \mathcal{Q}_n[\Psi](M) \mid \ \ \forall\, i = 1, \dots, n, \ \forall\, \mu \in M \ \ \lambda_i(A + Q, \mu) \geqslant \lambda_i(A) \Big\}.$$

The class $\mathcal{Q}_n[\Psi, A](M)$ is nonempty for the same reasons as class (3) is.

The problem is to describe the set of all pairs $(\Lambda(A), \Lambda(A + Q, \,\cdot\,))$, where $A \in \widetilde{\mathcal{M}}_n$ and $Q \in \mathcal{Q}_n[\Psi, A](M)$, i.e. the set

$$\Pi\mathcal{Q}_n[\Psi](M) = \Big\{ (\Lambda(A), \Lambda(A + Q, \,\cdot\,)) \mid \ \ A \in \widetilde{\mathcal{M}}_n, \ Q \in \mathcal{Q}_n[\Psi, A](M) \Big\},$$

for given $n \in \mathbb{N}$, metric space M, and set $\Psi \subset \Phi$.

The solution to this problem for a countable set Ψ is stated in the following

Theorem 2. *For any metric space M, $n \geqslant 2$ and countable set $\Psi \subset \Phi$ the pair $(l, f(\,\cdot\,))$, where $l \in (\overline{\mathbb{R}})^n$ and $f(\,\cdot\,) : M \to (\overline{\mathbb{R}})^n$, belongs to the set $\Pi\mathcal{Q}_n[\Psi](M)$ if and only if conditions* 1)–4) *of Theorem 1 are met.*

The last result shows that all theoretically possible pairs of the spectrum of an original and parametrically perturbed systems (with an additional condition that all the exponents of a perturbed system are not less than those of the original one) can be obtained even in the class of perturbations that decay arbitrarily fast at infinity. This situation is specific for systems with unbounded coefficients since the Lyapunov exponents of a system with bounded coefficients are invariant under perturbations that decay faster than any exponent [4, § 8.1].

References

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