Conditional Partial Integrals of Polynomial Hamiltonian Systems

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1 Introduction

Consider a canonical Hamiltonian odinary differential system with n degrees of freedom

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \quad \frac{dp_i}{dt} = -\partial_{q_i} H(q,p), \ \ i = 1, \dots, n, \tag{1.1}$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, $t \in \mathbb{R}$, and the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a polynomial of degree $h \geqslant 2$.

In this paper, using the Darboux theory of integrability [4] and the notion of conditional partial integral [5,7], we will study the existence of additional first integrals of the Hamiltonian system (1.1).

The Darboux theory of integrability (or the theory of partial integrals) was established by the French mathematician *Jean-Gaston Darboux* [4] in 1878, which provided a link between the existence of first integrals and invariant algebraic curves (or partial integrals) for polynomial autonomous differential systems. For the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see, for example, [5, 6, 11, 14, 15]). Note that the Darboux theory of integrability is related to the Poincaré problem [13], which asks to find the upper bound of invariant algebraic curves of planar polynomial differential systems. The Darboux theory of integrability is also involved in the study of Hilbert's 16-th problem (see, for example, the paper by Yu. Ilyashenko [9]). For the current state of the theory of integrability of differential systems see the monographs [2, 6, 8, 10, 11, 14, 15] and the references therein.

To avoid ambiguity, we give the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(G)$ on a domain $G \subset \mathbb{R}^{2n}$ is the function

$$\big[u(q,p), v(q,p)\big] = \sum_{i=1}^{n} \Big(\partial_{q_i} u(q,p)\, \partial_{p_i} v(q,p) - \partial_{p_i} u(q,p)\, \partial_{q_i} v(q,p)\Big) \ \text{ for all } \ (q,p) \in G.$$

A function $F \in C^1(G)$ is called a *first integral on the domain* G of the Hamiltonian system (1.1) if the functions F and H are in involution, i.e.,

$$\big[F(q,p), H(q,p)\big] = 0 \ \text{ for all } \ (q,p) \in G \subset \mathbb{R}^{2n}.$$

The Hamiltonian differential system (1.1) is *completely integrable* (in the Liouville sense) if it has n functionally independent first integrals which are in involution. Notice that the Hamiltonian H is a first integral of the Hamiltonian differential system (1.1).

A set of functionally independent on a domain $G \subset \mathbb{R}^{2n}$ first integrals $F_l \in C^1(G)$, $l = 1, \dots, k$, of the Hamiltonian system (1.1) is called a *basis of first integrals* (or *integral basis*) on the domain G of system (1.1) if any first integral $F \in C^1(G)$ of system (1.1) can be represented on G in the form

$$F(q,p) = \Phi\big(F_1(q,p), \dots, F_k(q,p)\big) \ \text{ for all } \ (q,p) \in G,$$

where Φ is some continuously differentiable function. The number k is said to be the *dimension* of basis of first integrals on the domain G for the Hamiltonian differential system (1.1).

The autonomous Hamiltonian differential system (1.1) on a domain without equilibrium points has an integral basis (autonomous) of dimension $2n - 1$ [1, pp. 167 – 169].

A real polynomial w is a *partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[w(q,p), H(q,p)\big] = w(q,p)M(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n},$$

where the polynomial M (*cofactor* of the partial integral w) is such that $\deg M \leqslant h - 2$.

Suppose w be a partial integral of the Hamiltonian differential system (1.1). Then the algebraic hypersurface $\{(q,p) : w(q,p) = 0\}$ is invariant by the flow of the Hamiltonian differential system (1.1) and if the cofactor M of the partial integral w is zero, then w is a polynomial first integral.

An exponential function $\omega(q,p) = \exp v(q,p)$ for all $(q,p) \in \mathbb{R}^{2n}$ with some real polynomial v is called a *conditional partial integral* of the Hamiltonian system (1.1) if the Poisson bracket

$$\big[v(q,p), H(q,p)\big] = S(q,p) \ \text{ for all } \ (q,p) \in \mathbb{C}^{2n},$$

where the polynomial S (*cofactor* of the conditional partial integral ω) is such that $\deg S \leqslant h - 2$.

We stress that a conditional partial integral is a special case of exponential factor (or exponential partial integral) [3, 5, 11] for the polynomial Hamiltonian ordinary differential system (1.1).

2 Main results

Suppose the Hamiltonian differential system (1.1) has real partial integrals w_l with the cofactors M_l, $l = 1, \dots, s$, respectively, such that the Poisson brackets

$$\big[w_l(q,p), H(q,p)\big] = w_l(q,p)M_l(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \quad \deg M_l \leqslant h - 2, \ \ l = 1, \dots, s. \tag{2.1}$$

And moreover, the polynomial Hamiltonian system (1.1) has conditional partial integrals

$$\omega_\nu(q,p) = \exp v_\nu(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \ \ \nu = 1, \dots, m, \tag{2.2}$$

with polynomials v_ν, $\nu = 1, \dots, m$, such that the following identities hold

$$\big[v_\nu(q,p), H(q,p)\big] = S_\nu(q,p) \ \text{ for all } \ (q,p) \in \mathbb{R}^{2n}, \quad \deg S_\nu \leqslant h - 2, \ \ \nu = 1, \dots, m. \tag{2.3}$$

Theorem 2.1. *Let the exponential functions* (2.2) *be conditional partial integrals of the polynomial Hamiltonian differential system* (1.1). *Then the scalar function*

$$F(q,p) = \sum_{\nu=1}^{m} \beta_\nu v_\nu(q,p) \ \textit{ for all } \ (q,p) \in \mathbb{R}^{2n}, \ \ \beta_\nu \in \mathbb{R}, \ \ \nu = 1, \dots, m, \ \ \sum_{\nu=1}^{m} |\beta_\nu| \neq 0, \tag{2.4}$$

is an additional first integral of the Hamiltonian system (1.1) *if and only if*

$$\sum_{\nu=1}^{m} \beta_\nu S_\nu(q,p) = 0 \ \textit{ for all } \ (q,p) \in \mathbb{R}^{2n}. \tag{2.5}$$

Proof. Taking into account the identities (2.3) and bilinearity of Poisson brackets, we calculate the Poisson bracket of the function (2.4) and the Hamiltonian H:

$$\big[F(q,p),H(q,p)\big]=\Big[\sum_{\nu=1}^{m}\beta_\nu v_\nu(q,p),H(q,p)\Big]=\sum_{\nu=1}^{m}\beta_\nu\big[v_\nu(q,p),H(q,p)\big]=\sum_{\nu=1}^{m}\beta_\nu S_\nu(q,p).$$

Therefore, by definition of first integral, the function (2.4) is a first integral of the polynomial Hamiltonian ordinary differential system (1.1) if and only if the identity (2.5) is true. □

Theorem 2.2. *Suppose the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *such that the identities* (2.3) *are true under the conditions*

$$S_\nu(q,p)=\mu_\nu M(q,p)\ \textit{ for all }\ (q,p)\in\mathbb{R}^{2n},\ \ \mu_\nu\in\mathbb{R},\ \ \nu=1,\dots,m,\ \ \deg M\leqslant h-2. \tag{2.6}$$

Then the scalar function (2.4) *is an additional first integral of the Hamiltonian differential system* (1.1) *if real numbers* β_ν *are a solution to the linear equation* $\sum\limits_{\nu=1}^{m}\mu_\nu\beta_\nu=0$ *under* $\sum\limits_{\nu=1}^{m}|\beta_\nu|\neq 0$.

Proof. If the representations (2.6) are true and numbers β_ν are a solution to $\sum\limits_{\nu=1}^{m}\mu_\nu\beta_\nu=0$, then

$$\sum_{\nu=1}^{m}\beta_\nu S_\nu(q,p)=\sum_{\nu=1}^{m}\beta_\nu\mu_\nu M(q,p)=0.$$

This implies that the condition (2.5) is true. Therefore, by Theorem 2.1, the function (2.4) is an additional first integral of the Hamiltonian system (1.1). □

From Theorem 2.2 under $m=2$, $\mu_1=\mu_2\neq 0$, we get the following statement.

Corollary 2.1. *If the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *under the condition* $m=2$ *such that the identity holds*

$$\frac{[v_1(q,p),H(q,p)]}{[v_2(q,p),H(q,p)]}=\frac{v_1(q,p)}{v_2(q,p)}\ \textit{ for all }\ (q,p)\in G\subset\mathbb{R}^{2n},$$

then an additional first integral of the polynomial Hamiltonian system (1.1) *is the function*

$$F:(q,p)\to\ v_1(q,p)-v_2(q,p)\ \textit{ for all }\ (q,p)\in\mathbb{R}^{2n}.$$

From Theorem 2.2 under $m=2$, $\mu_1=\,-\,\mu_2\neq 0$, we obtain the following statement.

Corollary 2.2. *If the polynomial Hamiltonian differential system* (1.1) *has the conditional partial integrals* (2.2) *under the condition* $m=2$ *such that the identity holds*

$$\frac{[v_1(q,p),H(q,p)]}{[v_2(q,p),H(q,p)]}=-\frac{v_1(q,p)}{v_2(q,p)}\ \textit{ for all }\ (q,p)\in G\subset\mathbb{R}^{2n},$$

then an additional first integral of the polynomial Hamiltonian system (1.1) *is the function*

$$F:(q,p)\to\ v_1(q,p)+v_2(q,p)\ \textit{ for all }\ (q,p)\in\mathbb{R}^{2n}.$$

Corollary 2.3. *Under the conditions of Theorem 2.2, we get the scalar functions*

$$F_{\xi\zeta}(q,p)=\beta_\xi v_\xi(q,p)+\beta_\zeta v_\zeta(q,p) \ \textit{ for all } \ (q,p)\in\mathbb{R}^{2n}, \ \ \xi,\zeta=1,\dots,m, \ \ \zeta\neq\xi,$$

are first integrals of the polynomial Hamiltonian system (1.1)*, where numbers* β_ξ *and* β_ζ *are solutions to the linear homogeneous equations*

$$\mu_\xi\beta_\xi+\mu_\zeta\beta_\zeta=0$$

under

$$|\beta_\xi|+|\beta_\zeta|\neq 0, \ \ \xi,\zeta=1,\dots,m, \ \ \zeta\neq\xi.$$

Theorem 2.3. *Suppose the Hamiltonian system* (1.1) *has the partial integrals* w_l, $l=1,\dots,s$, *such that the identities* (2.1) *hold with* $M_l(q,p)=\lambda_l M(q,p)$ *for all* $(q,p)\in\mathbb{R}^{2n}$, $\lambda_l\in\mathbb{R}$, $l=1,\dots,s$, *and the conditional partial integrals* (2.2) *such that the identities* (2.3) *under* (2.6) *are true. Then*

$$F_{\xi\zeta}(q,p)=w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big) \ \textit{ for all } \ (q,p)\in G\subset\mathbb{R}^{2n}, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m, \tag{2.7}$$

are first integrals of system (1.1)*, where numbers* γ_ξ *and* β_ζ *are solutions to the equations*

$$\lambda_\xi\gamma_\xi+\mu_\zeta\beta_\zeta=0 \ \textit{ under the conditions } \ |\gamma_\xi|+|\beta_\zeta|\neq 0, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m. \tag{2.8}$$

Proof. Using the functional identities (2.1) and (2.3), we obtain

$$\begin{aligned}
&\big[F_{\xi\zeta}(q,p),H(q,p)\big]\\
&\qquad=\big[w_\xi^{\gamma_\xi}(q,p),H(q,p)\big]\cdot\exp\big(\beta_\zeta v_\zeta(q,p)\big)+w_\xi^{\gamma_\xi}(q,p)\cdot\big[\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big]\\
&\qquad=\gamma_\xi w_\xi^{\gamma_\xi-1}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big[w_\xi(q,p),H(q,p)\big]\\
&\qquad\qquad+\beta_\zeta w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\big[v_\zeta(q,p),H(q,p)\big]\\
&\qquad=\big(\lambda_\xi!\gamma_\xi+\mu_\zeta!\beta_\zeta\big)M(q,p)w_\xi^{\gamma_\xi}(q,p)\exp\big(\beta_\zeta v_\zeta(q,p)\big)\\
&\qquad\qquad\qquad\text{for all } \ (q,p)\in G, \ \ \xi=1,\dots,s, \ \ \zeta=1,\dots,m.
\end{aligned}$$

If the real numbers γ_ξ and β_ζ are solutions to the linear equations (2.8), then the functions (2.7) are additional first integrals of the polynomial Hamiltonian differential system (1.1). ☐

For example, the polynomial Hamiltonian differential system given by [12]

$$H(q,p)=\frac{1}{2}\,(p_1^2+p_2^2)+2q_2p_1p_2-q_1 \ \text{ for all } \ (q,p)\in\mathbb{R}^4 \tag{2.9}$$

has the polynomial partial integral $w(q,p)=p_2$ with cofactor $M(q,p)=-2p_1$ and the conditional partial integral $\omega(q,p)=\exp p_1^2$ with cofactor $S(q,p)=2p_1$. By Theorem 2.3, we can build the additional first integral of the Hamiltonian system (2.9): $F(q,p)=p_2\exp p_1^2$ for all $(q,p)\in\mathbb{R}^4$. The functionally independent first integrals H and F of the Hamiltonian system (2.9) are in involution. Therefore, the Hamiltonian system (2.9) is completely integrable (in the Liouville sense).

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