Robust Stability of Global Attractors for Reaction-Diffusion System w.r.t. Disturbances

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1 Introduction

Asymptotic stability of an equilibrium is a fundamental property of evolutionary processes and plays important role for many applications. It is well-known that a globally asymptotically stable equilibrium of a linear and finite dimensional system is robust in the sense that for any essentially bounded external disturbance entering to the system the corresponding solution remains bounded for all times and that it tends to a ball around the equilibrium when time goes to infinity. The size of this ball depends on the disturbance norm only. For nonlinear systems this is in general not true and leads to the notion of Input-to-State Stability (ISS), introduced by E. D. Sontag [12, 13] for finite dimensional systems. This notion is also suitable to study robustness of equilibria in case of infinite dimensional systems [2]. During the last decade many authors tried to extend the ISS theory to this class of systems. Many of these extensions were developed for systems given in terms of partial differential equations (PDEs), see, for example, the works [4, 5, 7, 11]. It should be noted that almost all ISS-like results for PDEs were developed for the case of single equilibrium point of the unperturbed system. It is well known that many nonlinear systems possess a nontrivial global attractor instead. In this work we study the question of robustness of such an attracting set with respect to external disturbances. Existence and different properties of global attractors were studied in many books [1,10,14] and papers [3,6]. We are interested in the following question: given a system possessing a global attractor, what can we say about attracting sets for solutions if some perturbation h enters to this system? In this work, we consider such a question for perturbed reaction-diffusion system. Using a general scheme suggested in [4], we prove local ISS and asymptotic gain (AG) properties w.r.t. global attractors for dissipative RD system.

2 Statement of the problem and the main results

In a bounded domain $\Omega \subset \mathbb{R}^n$, we consider the following parabolic problem (named Reaction-Diffusion system)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x) + d(t,x), \quad x \in \Omega, \ \ t > 0, \\ u\big|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $u = u(t,x) = (u^1(t,x), \dots, u^N(t,x))$ is an unknown vector-function, $f = (f^1, \dots, f^N)$, $h = (h^1, \dots, h^N)$ are given functions, a is a real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \mu I$, $\mu > 0$, $d = (d^1, \dots, d^N)$ is an external disturbances.

We assume that the following properties hold:

$$h \in (L^2(\Omega))^N, \ \ f \in C^1(\mathbb{R}^N;\mathbb{R}^N),$$

$\exists\, C_1, C_2 > 0$, $C_3 > 0$, $\gamma_i > 0$, $p_i \geq 2$, $i = 1, \dots, N$ such that $\forall\, v \in \mathbb{R}^N$

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i-1}} \leq C_1\Big(1 + \sum_{i=1}^N |v^i|^{p_i}\Big), \quad \sum_{i=1}^N f^i(v)v^i \geq \sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_2,$$
$$\forall\, w \in \mathbb{R}^N \ \ (Df(v)w, w) \geq -C_3 \sum_{i=1}^N |w^i|^2.$$

In further arguments we will use standard functional spaces

$$H = (L^2(\Omega))^N \ \text{ and } \ V = (H_0^1(\Omega))^N.$$

Let us denote

$$p = (p_1, \dots, p_N), \ \ L^p(\Omega) = L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega).$$

It is known [1] that under such assumptions for every disturbances $d \in L^\infty(\mathbb{R}_+; (L^2(\Omega))^N)$ the problem (2.1) is globally uniquely resolvable in a weak sense in the phase space H, i.e., for every $u_0 \in H$ there exists a unique function $u = u(t,x) \in L^2_{loc}(0,+\infty; V) \cap L^p_{loc}(0,+\infty; L^p(\Omega))$ such that for all $T > 0$, $v \in V \cap L^p(\Omega)$,

$$\frac{d}{dt}\int\limits_\Omega u(t,x)v(x)\,dx + \int\limits_\Omega \Big(a\nabla u(t,x)\nabla v(x) + f(u(t,x))v(x) - h(x)v(x) - d(t,x)v(x)\Big)\,dx = 0$$

in the sense of scalar distributions on $(0,T)$, and $u(0,x) = u_0(x)$.

Due to the inclusion $u \in C([0,+\infty); H)$, the last equality makes sense.

Let us consider the unperturbed system ($d \equiv 0$)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x), & x \in \Omega, \ \ t > 0, \\ u\big|_{\partial\Omega} = 0. \end{cases} \tag{2.2}$$

It is known [1] that the corresponding semigroup $S : \mathbb{R}_+ \times H \mapsto H$

$$S(t, u_0) = u(t), \ \text{ where } u(\,\cdot\,) \text{ is a global weak solution of (2.2)}, \ \ u(0) = u_0$$

possesses a global attractor Θ in H, i.e., there exists a compact set $\Theta \subset H$ such that the following properties hold:

(i) $\Theta = S(t, \Theta)$, $t \geq 0$;

(ii) for any bounded $B \subset H$

$$\operatorname{dist}(S(t,B), \Theta) \to 0 \ \text{ as } \ t \to \infty,$$

where for given $A, B \subset H$ we denote

$$\operatorname{dist}(A,B) = \sup_{x\in A}\inf_{y\in B} \|x-y\|_H.$$

This property guarantees that any solution to (2.2) approaches Θ as $t \to \infty$. We are interested in the long term behavior of the corresponding solutions in the case the system (2.1) is perturbed by external perturbation d.

Given an initial state $u_0 := u(0) \in H$ and a perturbing signal $d \in L^\infty(\mathbb{R}_+; H)$, the corresponding unique solution to (2.1) is denoted by $u(t, u_0, d)$. Due to the disturbance we have no guarantee, in general, that this solution will converge to Θ as $t \to \infty$. It turns out that the global attractor is robust under perturbation, i.e., its attractivity properties are affected only slightly by disturbances of small magnitude. This robustness property can be expressed in the ISS framework as follows: there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any $u_0 \in H$ and $d \in L^\infty(\mathbb{R}_+; H)$,

$$\|u(t, u_0, d)\|_\Theta \le \beta\big(\|u_0\|_\Theta, t\big) + \gamma\big(\|d\|_\infty\big), \quad t \ge 0, \tag{2.3}$$

where the well-known classes $\mathcal{K}$ stands for the class of continuous strictly increasing functions on $[0, +\infty)$ vanishing at the origin, $\mathcal{KL}$ is the set of continuous functions defined on $[0, +\infty)^2$ which are of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second one,

$$\|d\|_\infty = \operatorname*{ess\,sup}_{t \ge 0} \|d(t)\|_H,$$
$$\|u\|_\Theta = \inf_{\theta \in \Theta} \|u - \theta\|_H.$$

It should be noted that in the general case $\Theta \ne \{0\}$. Moreover, its structure is very complicated [6]. Therefore, estimates like (2.3) cannot be obtained by using direct a priori estimates.

In this paper we prove that local variant of this property holds for the problem (2.1).

Unfortunately, this property is in general not guaranteed even for the case $\Theta = \{0\}$, see, for example, [2].

In this paper we prove this property for the problem (2.1) at least locally. More precisely, we prove the following result

Theorem. *Under the mentioned above assumptions the problem* (2.1)

(i) *is local ISS with respect to* Θ*, i.e., there exists* $r > 0$*,* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *such that for any* $\|u_0\|_H \le r$ *and any* $\|d\|_\infty \le r$*,*

$$\|u(t, u_0, d)\|_\Theta \le \beta\big(\|u_0\|_\Theta, t\big) + \gamma\big(\|d\|_\infty\big), \quad t \ge 0; \tag{2.4}$$

(ii) *satisfies the asymptotic gain (AG) property with respect to* Θ*, that is there exists* $\gamma \in \mathcal{K}$ *such that for any* $u_0 \in H$ *and any* $d \in L^\infty(\mathbb{R}_+; H)$ *it holds*

$$\limsup_{t \to \infty} \|u(t, u_0, d)\|_\Theta \le \gamma\big(\|d\|_\infty\big). \tag{2.5}$$

To prove the local ISS property, the Laypunov technique is used. To establish the AG property, the uniform attractors theory for non-autonomous systems [1] is used.

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