Sign-indefinite Logistic Models with Flux-Saturated Diffusion: Existence and Multiplicity Results

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Let us consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda a(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the diffusion is driven by the mean curvature operator $-\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$. In equation (1), $\lambda > 0$ is a parameter measuring diffusivity and

(H_1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, with a C^2 boundary $\partial\Omega$ in case $N \ge 2$;

(H_2) $a : \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $\max\limits_{\overline{\Omega}} a > 0$;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying, for some constant $L > 0$, $f(0) = f(L) = 0$, and $f(s) > 0$ for every $s \in \,]0, L[\,$.

Assumption (H_2) on the weight a introduces spatial heterogeneities within the model and allows that a changes sign in Ω. Assumption (H_3) basically requires that the reaction term af is of logistic-type. As it is well-known, logistic maps play a pivotal role in the modeling theory of various disciplines, with special prominence in biology, ecology, genetics. Unlike the classical theory based on the Fick–Fourier's law, where the flux depends linearly on ∇u, here the diffusion is governed by the bounded flux $\nabla u/\sqrt{1+|\nabla u|^2}$, which is approximately linear for small gradients but approaches saturation for large ones.

Following our recent paper [3], we aim here to synthetically describe and clarify the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. This study is motivated by the investigations on reaction processes with saturating diffusion started in [4], in order to correct the non-physical gradient-flux relations at high gradients. This specific mechanism of diffusion, of which the mean curvature operator provides a paradigmatic example, may determine spatial patterns exhibiting abrupt transitions at the boundary or between adjacent profiles, up to the formation of discontinuities. This makes the mathematical analysis of the problem (1) more delicate and sophisticated than the study of the corresponding semilinear model, the use of some tools of geometric measure theory being in particular required. Indeed, it is an established fact that the space of bounded variation functions is the natural setting for dealing with this problem. The precise notion of bounded variation solution of (1) used in this paper has been basically introduced in [1] and is recalled below for completeness.

Notation. For every $v \in BV(\Omega)$, $Dv = D^a v\,\mathrm{d}x + D^s v$ is the Lebesgue–Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $D^a v\,\mathrm{d}x$ and its singular part $D^s v$ with respect to the N-dimensional Lebesgue measure $\mathrm{d}x$ in $\mathbb{R}^N$, $|Dv|$ denotes the total variation of the measure Dv, and $\frac{Dv}{|Dv|}$ stands for the density of Dv with respect to its total variation. Further, $|\Omega|$ is the Lebesgue measure of Ω, while $\mathcal{H}_{N-1}$ represents the $(N-1)$-dimensional Hausdorff measure, and $|\partial\Omega|$ is the $\mathcal{H}_{N-1}$-measure of $\partial\Omega$. Moreover, for all functions $u, v : \Omega \to \mathbb{R}$, we write: $u \geq v$ if $\operatorname{ess\,inf}(u - v) \geq 0$; $u > v$ if $u \geq v$ and $\operatorname{ess\,sup}(u - v) > 0$; $u \gg v$ if, for a.e. $x \in \Omega$, $u(x) - v(x) \geq \operatorname{dist}(x, \partial\Omega)$.

Definition. By a *bounded variation solution* of (1) we mean a function $u \in BV(\Omega)$, with $f(u) \in L^N(\Omega)$, which satisfies

$$\int\limits_{\Omega} \frac{D^a u\, D^a \phi}{\sqrt{1 + |D^a u|^2}}\,\mathrm{d}x + \int\limits_{\Omega} \frac{Du}{|Du|}\,\frac{D\phi}{|D\phi|}\,|D^s \phi| + \int\limits_{\partial\Omega} \operatorname{sgn}(u)\phi\,\mathrm{d}\mathcal{H}_{N-1} = \lambda \int\limits_{\Omega} a f(u)\phi\,\mathrm{d}x \tag{2}$$

for every $\phi \in BV(\Omega)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ and $\phi(x) = 0$ $\mathcal{H}_{N-1}$-a.e. on the set $\{x \in \partial\Omega :\ u(x) = 0\}$. A bounded variation solution u is said *positive* if $u > 0$.

Remark 1. If a bounded variation solution u of (1) belongs to $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for some $p > N$, then it satisfies the differential equation in (1) for a.e. $x \in \Omega$ and the boundary condition for all $x \in \partial\Omega$. Therefore, u is a *strong solution* of (1). The L^p-regularity theory then entails that $u \in W^{2,q}(\Omega)$ for all $q > N$. Conversely, it is evident that any strong solution is a bounded variation solution. Note that bounded variation solutions, unlike the strong ones, may not satisfy the Dirichlet boundary conditions.

Remark 2. It is clear that, for any given $\lambda > 0$, $u = 0$ is a bounded variation solution of (1), while $u = L$ is not. Indeed, if L were a solution, taking $\phi = 1$ as test function in (2) would yield $\int\limits_{\partial\Omega} 1\,\mathrm{d}\mathcal{H}_{N-1} = |\partial\Omega| = 0$, which is a contradiction.

We are now going to present the main results obtained in [3]. Here, for the sake of clarity, our statements are set out in a simplified form, while referring to [3] for some variants or extensions that rely on slightly more general but less neat conditions.

The first result only exploits the structural assumptions (H_1), (H_2), and (H_3). It provides us with the existence of a number $\lambda_* \geq 0$ such that, for all $\lambda > \lambda_*$, the problem (1) has a maximum solution u_λ, with $0 < u_\lambda < L$. The asymptotic behavior of u_λ, as $\lambda \to +\infty$, is described too, and the bifurcation of the solutions from the trivial line $\{(\lambda, 0) :\ \lambda \geq 0\}$ at the point $(0, 0)$ is ascertained in the case $\lambda_* = 0$. Figure 1 illustrates two admissible bifurcations diagrams.

Theorem 1. *Assume (H_1), (H_2) and (H_3). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in]\lambda_*, +\infty[$ the problem* (1) *admits a maximum bounded variation solution u_λ, with $0 < u_\lambda < L$, which satisfies*

$$\lim_{\lambda \to +\infty} (\operatorname{ess\,sup} u_\lambda) = L. \tag{3}$$

Moreover, if $\lambda_* = 0$, then

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{BV} = 0. \tag{4}$$

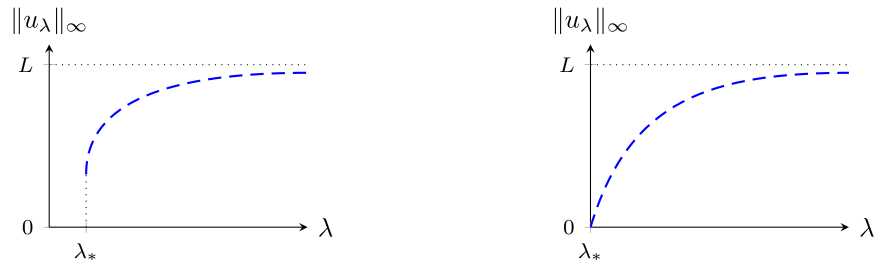

Figure 1. Admissible bifurcation diagrams for the problem (1) under the structural assumptions (H_1), (H_2), and (H_3), in case $\lambda_* > 0$ (left) or $\lambda_* = 0$ (right). Dashed curves indicate bounded variation solutions.

The specific features displayed by the bifurcation diagrams of the problem (1) are determined by the slope at 0 of the function f, as expressed by the following conditions:

(H_4) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = +\infty$ (*sublinear growth at* 0);

(H_5) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = \kappa \in]0, +\infty[$ (*linear growth at* 0);

(H_6) there exists $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = 0$ (*superlinear growth at* 0).

When f has a sublinear growth at zero, a bifurcation from the trivial line occurs at the point $(0, 0)$, and the existence of positive bounded variation solutions of the problem (1) is guaranteed for all $\lambda > 0$. In addition, positive strong solutions exist provided that λ is small enough.

Theorem 2. *Assume (H_1), (H_2), (H_3), and (H_4). Then for all $\lambda > 0$ the problem* (1) *admits at least one bounded variation solution $u_\lambda \in BV(\Omega)$, with $0 < u_\lambda < L$, which satisfies* (3) *and* (4). *Moreover, there exists $\lambda^* > 0$ such that, for all $\lambda \in]0, \lambda^*[$, solutions u_λ can be selected so that $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for any $p > N$, it is a strong solution and it satisfies*

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{W^{2,p}} = 0. \tag{5}$$

When f grows linearly at zero the bifurcation occurs from the trivial line at the point $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of the linear weighted problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\kappa\varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω satisfies (H_1), κ comes from (H_5), and a satisfies (H_2). It is a classical fact that λ_1 is positive and simple, with a positive eigenfunction φ_1. The L^p-regularity theory and a standard bootstrap argument entail that $\varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p > N$, while the strong maximum principle and the Hopf boundary point lemma yield $\varphi_1 \gg 0$. In this case the solvability of the problem (1) is guaranteed for all $\lambda > \lambda_1$. In addition, for λ close to λ_1 strong solutions do exist.

Theorem 3. *Assume (H_1), (H_2), (H_3), and (H_5). Then for all $\lambda > \lambda_1$ the problem* (1) *admits at least one bounded variation solution u_λ, with $0 < u_\lambda < L$, which satisfies* (3). *Moreover, suppose that*

(H_7) *f is of class C^2*

and fix any $p > N$. Then there exists a neighborhood $\mathcal{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that solutions u_λ can be selected so that $(\lambda, u_\lambda) \in \mathcal{U}$, u_λ is a strong solution and it satisfies

$$\lim_{\lambda \to \lambda_1} \|u_\lambda\|_{W^{2,p}} = 0 \ \ \text{and} \ \ \lim_{\lambda \to \lambda_1} \frac{u_\lambda}{\|u_\lambda\|_{C^1}} = \varphi_1. \tag{6}$$

Finally, there exists $\eta > 0$ such that the following assertions hold:

- (i) *if $f''(0) < 0$, then for all $\lambda \in]\lambda_1, \lambda_1 + \eta[$ there is at least one strong solution $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying* (6)*;*
- (ii) *if $f''(0) > 0$, then for all $\lambda \in]\lambda_1 - \eta, \lambda_1[$ there is at least one strong solution $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying* (6)*.*

Remark 3. For the standard logistic model $f(s) = s(L - s)$, the condition $f''(0) = -2 < 0$ holds and therefore the bifurcation is supercritical.

When f exhibits a superlinear growth at zero, the existence of multiple solutions can be detected if, for instance, conditions (H_2) and (H_6) are strengthened as follows. Let us set

$$\Omega^+ = \big\{x \in \Omega : \ a(x) > 0\big\}, \quad \Omega^- = \big\{x \in \Omega : \ a(x) < 0\big\}, \quad \Omega^0 = \big\{x \in \Omega : \ a(x) = 0\big\},$$

and replace (H_2) with

(H_8) $a \in C^2(\overline{\Omega})$, $\Omega^+ \neq \varnothing$, $\Omega^- \neq \varnothing$, $\Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega$, and $\nabla a(x) \neq 0$ for all $x \in \Omega^0$,

as well as (H_6) with

(H_9) there exists $q > 1$, with $q < \frac{N+2}{N-2}$ if $N \geq 3$, such that

$$\lim_{s \to 0^+} \frac{f(s)}{s^q} = 1.$$

Then, for λ sufficiently large, the problem (1) has at least two positive bounded variation solutions, the smaller being strong.

Theorem 4. *Assume (H_1), (H_3), (H_8), and (H_9). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in]\lambda_*, +\infty[$, the problem* (1) *admits at least one bounded variation solution u_λ and one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for any $p > N$, such that $0 \ll v_\lambda < u_\lambda < L$. In addition, u_λ satisfies* (3)*, while v_λ satisfies*

$$\lim_{\lambda \to +\infty} \|v_\lambda\|_{W^{2,p}} = 0. \tag{7}$$

Figure 2 illustrates three qualitatively different bifurcation diagrams corresponding, respectively, to Theorems 2, 3, and 4.

Unexpectedly enough, the existence of multiple solutions can always be detected in the standard logistic model, whenever the carrying capacity L is sufficiently large, even in the case where the weight function a is a positive constant (cf. Remark 4 below). We state such a multiplicity result for the simplest one-dimensional prototype of the problem (1), that is,

$$\begin{cases} -\bigg(\dfrac{u'}{\sqrt{1 + (u')^2}}\bigg)' = \lambda a f(u) \ \ \text{in} \ \ ]0, 1[\,, \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{8}$$

Theorem 5. *Assume (H_3),*

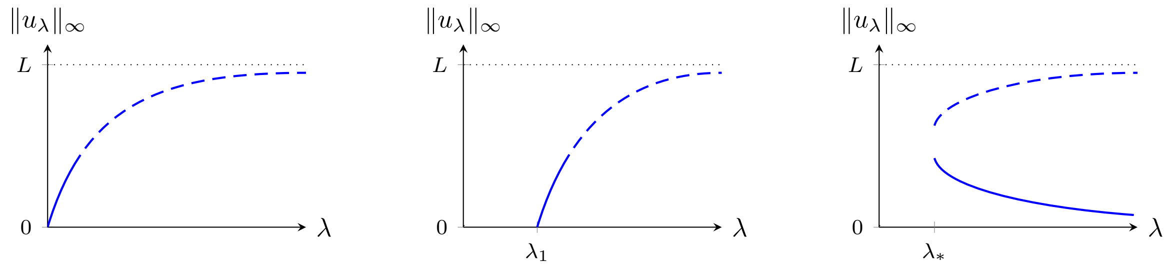

Figure 2. Admissible qualitative bifurcation diagrams for the problem (1), according to the growth of f at 0: either sublinear (left), or linear (center), or superlinear (right). Dashed curves indicate bounded variation solutions, solid curves represent strong solutions.

(H_{10}) *$a \in C^0([0,1])$ satisfies $a > 0$,*

and

(H_{11}) *there exist $r, R \in]0, L[$, with $r < R$, such that*

$$\frac{2F(r)}{r^2}\left(1+\sqrt{1+r^2}\right) < \frac{F(R)}{R}\,,$$

where $F(s) = \int\limits_0^s f(t)\,\mathrm{d}t$ is the potential of f. Then there exist $\lambda_\sharp$ and $\lambda^\sharp$, with $0 \le \lambda_\sharp < \lambda^\sharp$, such that for all $\lambda \in]\lambda_\sharp, \lambda^\sharp[$ the problem (8) admits at least two bounded variation solutions u_λ, v_λ such that $0 < u_\lambda < v_\lambda < L$.

It is worth stressing that the assumptions of Theorem 5 do not prevent f from being concave in $[0, L]$: this fact witnesses the peculiarity of this multiplicity result, which is specific of the quasilinear problem (1) and has no similarity with the semilinear case, where the concavity of f always guarantees the uniqueness of the positive solution, as proven in [2] even for sign-changing weights a.

Remark 4. For the standard logistic model, where $f(s) = s(L - s)$, condition (H_{11}) is satisfied if, for instance, $L > \frac{32}{3} \approx 10.67$.

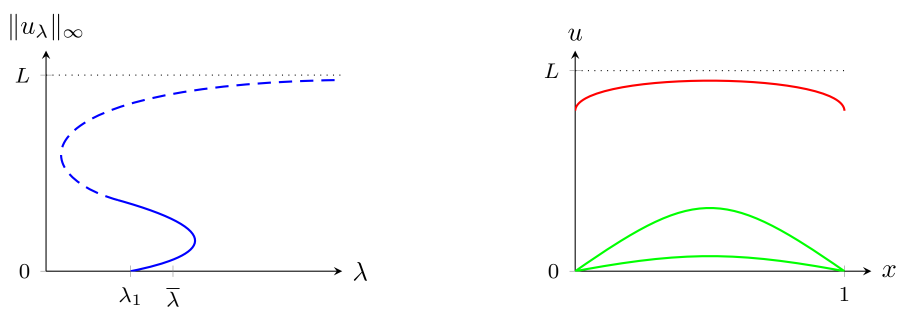

Figure 3. On the left, an admissible bifurcation diagram is depicted with reference to Example 1: the dashed curve indicates bounded variation solutions, the solid curve represents strong solutions. On the right, the profiles of the three detected solutions at $\lambda = \overline{\lambda}$ are shown: in green the regular ones, in red the singular one.

Example 1. A numerical study of the problem (8), with $a = 1$, $f(s) = s(L - s)$ and $L = 11 > \frac{32}{3}$, reveals the existence of three positive solutions in a (small) right neighborhood of the bifurcation point $\lambda_1 = \frac{\pi^2}{L} \approx 0.8972$, in particular at $\overline{\lambda} = 0.8975$, and of two positive solutions in a left neighborhood of λ_1. This is in complete agreement with (i) the bifurcation result stated in Theorem 3 and Remark 3, which predicts the bifurcation branch emanates from λ_1 pointing to the right; (ii) the multiplicity conclusions of Theorem 5, which guarantee the existence of two solutions in an interval of the λ-axis located on the left of λ_1. Hence an S-shaped bifurcation diagram is expected as shown by the picture on the left in Figure 3.

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