

Matrix Boundary-Value Problems for Differential Equations with p -Laplacian

Olga Nesmelova

*Institute of Applied Mathematics and Mechanics
of the National Academy of Sciences of Ukraine, Slavyansk, Ukraine
E-mail: star-o@ukr.net*

The boundary-value problems for differential equations with p -Laplacian arise while studying the radial solutions of nonlinear partial differential equations. A feature of such various boundary-value problems for differential, including difference equations with p -Laplacian is the lack of uniqueness of the solution.

In this thesis, we consider the boundary-value problem for the linear system of differential equations with matrix p -Laplacian, which is reduced to the traditional differential-algebraic system with an unknown in the form of the vector function. We considered two cases of the obtained differential-algebraic system, in particular, the cases of solvability and unsolvability of the differential-algebraic system with respect to the derivative. For both cases, we obtained a sufficient condition for the solvability of the matrix boundary-value problem for the differential equation with p -Laplacian, in which connection its general solution determines the general solution for the homogeneous part of the matrix differential equation with p -Laplacian and the Green operator of the original matrix boundary-value problem.

The relevance of studying the boundary-value problems for differential equations with p -Laplacian is associated with numerous applications of such problems in the theory of elasticity, the theory of plasma, and astrophysics. The purpose of this thesis is to generalize various boundary-value problems for differential equations with p -Laplacian, which preserves the features of the solution of such problems, namely, the lack of uniqueness of the solution, and, in this case, the dependence of the desired solution of the arbitrary function.

We have studied the problem on the construction of solutions [1–3, 5, 6]

$$Z(t) \in \mathbb{C}_{\alpha \times \beta}^2[a, b] := \mathbb{C}^2[a, b] \otimes \mathbb{R}^{\alpha \times \beta}$$

of the linear system of differential equations

$$\mathcal{P}Z(t) = A(t)Z(t) + F(t) \tag{1}$$

with the boundary condition

$$\mathcal{L}Z(\cdot) = \mathcal{A}, \quad \mathcal{A} \in \mathbb{R}^{\lambda \times \mu} \tag{2}$$

and with a matrix p -Laplacian $\mathcal{P}Z(t) := ((R(t)Z(t))'S(t))'$. Here,

$$R(t) \in \mathbb{C}_{\gamma \times \alpha}^2[a, b] := \mathbb{C}^2[a, b] \otimes \mathbb{R}^{\gamma \times \alpha}, \quad S(t) \in \mathbb{C}_{\beta \times \delta}^2[a, b] := \mathbb{C}^2[a, b] \otimes \mathbb{R}^{\beta \times \delta}, \\ F(t) \in \mathbb{C}_{\gamma \times \delta}[a, b] := \mathbb{C}^1[a, b] \otimes \mathbb{R}^{\gamma \times \delta},$$

$\mathcal{L}Z(\cdot)$ — is a linear bounded matrix functional: $\mathcal{L}Z(\cdot) : \mathbb{C}^2[a, b] \rightarrow \mathbb{R}^{\lambda \times \mu}$. Generally speaking, we assume that $\alpha \neq \beta \neq \gamma \neq \delta \neq \lambda \neq \mu$ are any natural numbers. By $\Xi^{(j)} \in \mathbb{R}^{\alpha \times \beta}$, $j = 1, 2, \dots, \alpha \cdot \beta$ we denote the natural basis of the space $\mathbb{R}^{\alpha \times \beta}$. In this case, the problem of determination of solutions

of equation (1) can be reduced to a problem of determination of a vector $z(t) \in \mathbb{C}_{\alpha\beta}^2[a; b]$, whose components $z_j(t) \in \mathbb{C}^2[a; b]$ define the expansion of the matrix

$$Z(t) = \sum_{j=1}^{\alpha\beta} \Xi^{(j)} z_j(t), \quad z_j(t) \in \mathbb{C}^1[a; b], \quad j = 1, 2, \dots, \alpha \cdot \beta$$

in vectors $\Xi^{(j)} \in \mathbb{R}^{\alpha \times \beta}$ of the basis of the space $\mathbb{R}^{\alpha \times \beta}$. We now define the operator

$$\mathcal{M}[\mathcal{A}] : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \cdot n}$$

as an operator that puts the matrix $\mathcal{A} \in \mathbb{R}^{m \times n}$ in correspondence to a vector-column $\mathcal{B} := \mathcal{M}[\mathcal{A}] \in \mathbb{R}^{m \cdot n}$ formed from n columns of the matrix A . We also introduce the inverse operator

$$\mathcal{M}^{-1}[\mathcal{B}] : \mathbb{R}^{m \cdot n} \rightarrow \mathbb{R}^{m \times n}$$

that puts the vector $\mathcal{B} \in \mathbb{R}^{m \cdot n}$ in correspondence to a matrix $\mathcal{A} \in \mathbb{R}^{m \times n}$. We present the product $A(t)Z(t)$ in the form

$$A(t)Z(t) := A(t) \sum_{k=1}^{\alpha\beta} \Xi^{(k)} z_k(t), \quad \mathcal{M}[A(t)Z(t)] = \check{A}(t) \cdot z(t),$$

where

$$\check{A}(t) := [\check{A}_k(t)]_{k=1}^{\alpha\beta} \in \mathbb{C}_{\gamma\delta \times \alpha\beta}[a, b], \quad \check{A}_k(t) = \mathcal{M}[A(t)\Xi^{(k)}], \quad k = 1, 2, \dots, \alpha \cdot \beta.$$

We now define the matrices

$$B(t), C(t), D(t) \in \mathbb{C}_{\gamma\delta \times \alpha\beta}[a, b]$$

in the following way:

$$\frac{\partial}{\partial z''} \mathcal{M}PZ(t) := B(t)z(t), \quad \frac{\partial}{\partial z'} \mathcal{M}PZ(t) := C(t)z(t), \quad \frac{\partial}{\partial z} \mathcal{M}PZ(t) := D(t)z(t).$$

The problem of construction of solutions of Eq. (1) can be reduced to a problem of determination of the vector $z(t) \in \mathbb{C}_{\alpha\beta}^2[a; b]$ that is defined by the system

$$B(t)z'' + C(t)z' + D(t)z = \check{A}(t)z + f(t), \quad f(t) := \mathcal{M}F(t).$$

Changing the variables

$$y_1 := z, \quad y_2 := y_1',$$

we get the problem of determination of the vector

$$y(t) \in \mathbb{C}_{2\alpha\beta}^2[a; b]$$

defined by the differential-algebraic system of equations [2, 3, 5]

$$U(t)y' = V(t)y + \check{f}(t), \tag{3}$$

where

$$U(t) := \begin{pmatrix} I_{\alpha\beta} & O_{\alpha\beta} \\ C(t) & B(t) \end{pmatrix}, \quad V(t) := \begin{pmatrix} O_{\alpha\beta} & I_{\alpha\beta} \\ \check{A}(t) - D(t) & O_{\gamma\delta \times \alpha\beta} \end{pmatrix}, \quad \check{f}(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Thus, under the condition [2, 3, 5]

$$P_{U^*(t)}V(t) = 0, \quad P_{U^*(t)}\check{f}(t) = 0, \tag{4}$$

we have proved the sufficient condition of solvability of the Cauchy problem for the matrix differential equation with p -Laplacian (1).

Lemma. Under conditions (4) the matrix Cauchy problem $Z(a) = \mathfrak{A}$ for the matrix differential equation with p -Laplacian (1) is uniquely solvable for any initial value of $\mathfrak{A} \in \mathbb{R}^{\mu \times \nu}$. Under conditions (4), the general solution

$$\begin{aligned} Z(t, c) &= W(t, c) + \mathcal{K}[\mathfrak{F}(s, \varphi(s))](t), \quad c \in \mathbb{R}^{2\alpha\beta}, \\ W(t, c) &:= \mathcal{M}^{-1}[\mathcal{J}_{\alpha\beta} X(t)c], \quad \mathcal{J}_{\alpha\beta} := \begin{bmatrix} I_{\alpha\beta} & O_{\alpha\beta} \end{bmatrix} \in \mathbb{R}^{\alpha\beta \times 2\alpha\beta}, \\ \mathcal{K}[\mathfrak{F}(s, \varphi(s))](t) &:= \mathcal{M}^{-1}\{\mathcal{J}_{\alpha\beta} K[\mathfrak{F}(s, \varphi(s))](t)\}, \end{aligned}$$

of the Cauchy problem

$$Z(a) = \mathfrak{A}$$

for the matrix differential equation with p -Laplacian (1) defines a generalized Green's operator $\mathcal{K}[\mathfrak{F}(s, \varphi(s))](t)$ of the Cauchy problem $Z(a) = 0$ for the matrix differential equation with p -Laplacian (1) and the general solution $W(t, c)$ of the Cauchy problem $Z(a) = \mathfrak{A}$ or the homogeneous part of the matrix differential equation with p -Laplacian (1).

Thus, in the critical case under conditions (4) and in the case of fulfillment of the solvability condition

$$P_{\mathcal{Q}^*} \mathcal{M}\{\mathfrak{A} - \mathcal{L}\mathcal{K}[\mathfrak{F}(s, \varphi(s))](\cdot)\} = 0 \tag{5}$$

the solution of the matrix boundary-value problem with p -Laplacian (1), (2) takes the form

$$Z(t, c_r) = W(t, c_r) + G[\mathfrak{F}(s, \varphi(s)); \mathfrak{A}](t), \quad W(t, c_r) := \mathcal{M}^{-1}[\mathcal{J}_{\alpha\beta} X(t)P_{\mathcal{Q}_r} c_r], \tag{6}$$

where

$$G[\mathfrak{F}(s, \varphi(s)); \mathfrak{A}](t) := \mathcal{M}^{-1}\left\{\mathcal{J}_{\alpha\beta} X(t)\mathcal{Q}^+ \mathcal{M}\{\mathfrak{A} - \mathcal{L}\mathcal{K}[\mathfrak{F}(s, \varphi(s))](\cdot)\}\right\} + \mathcal{K}[\mathfrak{F}(s, \varphi(s))](t).$$

Hence, we have proved the sufficient condition of solvability of the matrix boundary-value problem for the differential equation with p -Laplacian (1), (2).

Theorem. In the critical case ($P_{\mathcal{Q}^*} \neq 0$) under conditions (4) and (5), solution (6) of the matrix boundary-value problem with p -Laplacian (1), (2) determines the generalized Green's operator $G[\mathfrak{F}(s, \varphi(s)); \mathfrak{A}](t)$ of the matrix boundary-value problem with p -Laplacian (1), (2) and the general solution $W(t, c_r)$ for the homogeneous part of the differential equation with p -Laplacian (1), (2).

Assume that condition (4) is not satisfied [2, 3, 5], i.e., $P_{U^*(t)}V(t) \neq 0$ or $P_{U^*(t)}\check{f}(t) \neq 0$. Then the problem of determination of solutions of Eq. (1) after the change of the variables

$$y_1 := z := \Omega x_1, \quad y_2 := y'_1 := \Omega x'_1 := \Omega x_2, \quad \Omega \in \mathbb{R}^{\alpha\beta \times \alpha\beta}, \quad x(t) := \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

leads to the problem of determination of the vector $x(t) \in \mathbb{C}_{\alpha\beta}^2[a; b]$ defined by the differential-algebraic system of equations [2, 3, 5]

$$\check{U}(t)x' = \check{V}(t)x + \check{f}(t); \tag{7}$$

here,

$$\check{U}(t) := \begin{pmatrix} \Omega & O_{\alpha\beta} \\ C(t)\Omega & B(t)\Omega \end{pmatrix}, \quad \check{V}(t) := \begin{pmatrix} O_{\alpha\beta} & \Omega \\ (\check{A}(t) - D(t))\Omega & O_{\gamma\delta \times \alpha\beta} \end{pmatrix}.$$

Under the conditions [2, 3, 5]

$$P_{U^*(t)}\check{V}(t) = 0, \quad P_{U^*(t)}\check{f}(t) = 0, \tag{8}$$

system (7) is solvable with respect to the derivative [2, 3, 5]:

$$x' = \check{W}(t)x + \check{\mathfrak{F}}_1(t, \varphi(t)), \tag{9}$$

here,

$$\check{W}(t) := \check{U}^+(t)\check{V}(t), \quad \check{\mathfrak{F}}_1(t, \varphi(t)) := \check{U}^+(t)\check{f}(t) + P_{\check{U}_e}(t)\varphi(t).$$

Thus, we have proved the sufficient condition of solvability of the matrix Cauchy problem for the differential equation with p -Laplacian (1).

Corollary 1. *Under conditions (8), the matrix Cauchy problem $Z(a) = \mathfrak{A}$ for the matrix differential equation with p -Laplacian (1) is uniquely solvable for any initial value of $\mathfrak{A} \in \mathbb{R}^{\mu \times \nu}$. Under conditions (8), the general solution*

$$Z(t, c) = \check{W}(t, c) + \mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](t), \quad c \in \mathbb{R}^{2\alpha\beta},$$

$$\check{W}(t, c) := \mathcal{M}^{-1}[\Omega \mathcal{J}_{\alpha\beta} \check{X}(t)c], \quad \mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](t) := \mathcal{M}^{-1}\{\mathcal{J}_{\alpha\beta} K[\Omega \check{\mathfrak{F}}_1(s, \varphi(s))](t)\}.$$

of the Cauchy problem $Z(a) = \mathfrak{A}$ for the matrix differential equation with p -Laplacian (1) determines the generalized Green's operator $\mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](t)$ of the Cauchy problem $Z(a) = 0$ for the matrix differential equation with p -Laplacian (1) and the general solution $\check{W}(t, c)$ of the Cauchy problem $Z(a) = \mathfrak{A}$ for the homogeneous part of the matrix differential equation with p -Laplacian (1).

Hence, in the critical case under conditions (8) and in case of fulfillment of the condition of solvability

$$P_{\mathcal{Q}_d^*} \mathcal{M}\{\mathfrak{A} - \mathcal{L}\mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](\cdot)\} = 0, \tag{10}$$

the solution of the matrix boundary-value problem with p -Laplacian (1), (2) takes the form

$$Z(t, c_r) = \check{W}(t, c_r) + G[\check{\mathfrak{F}}_1(s, \varphi(s)); \mathfrak{A}](t), \quad \check{W}(t, c_r) := \mathcal{M}^{-1}[\Omega \mathcal{J}_{\alpha\beta} \check{X}(t)P_{\mathcal{Q}_r} c_r], \tag{11}$$

where

$$G[\check{\mathfrak{F}}_1(s, \varphi(s)); \mathfrak{A}](t) := \mathcal{M}^{-1}\left\{\Omega \mathcal{J}_{\alpha\beta} \check{X}(t)\mathcal{Q}^+ \mathcal{M}\{\mathfrak{A} - \mathcal{L}\mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](\cdot)\}\right\} + \mathcal{K}[\check{\mathfrak{F}}_1(s, \varphi(s))](t).$$

Thus, we have proved the sufficient condition of solvability of the matrix boundary-value problem for differential equation with p -Laplacian (1), (2).

Corollary 2. *In the critical case ($P_{\mathcal{Q}^*} \neq 0$) under conditions (8) and (10), solution (11) of the matrix boundary-value problem with p -Laplacian (1), (2) determines the generalized Green's operator $G[\check{\mathfrak{F}}_1(s, \varphi(s)); \mathfrak{A}](t)$ of the matrix boundary-value problem with p -Laplacian (1), (2) and the general solution $\check{W}(t, c_r)$ for the homogeneous part of the differential equation with p -Laplacian (1), (2).*

The research scheme proposed in the thesis can be transferred on the nonlinear matrix boundary value problems for differential equations with p -Laplacian, on the linear matrix boundary-value problems for difference equations, and on the matrix boundary-value problems for functional differential equations with p -Laplacian in abstract spaces, in particular, on the matrix boundary-value problems for differential equations with argument deviation. The proposed scheme of research of the linear system of differential equations with matrix p -Laplacian in the article was illustrated in details with examples.

References

- [1] S. L. Campbell, *Singular Systems of Differential Equations*. Research Notes in Mathematics, 40. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.
- [2] S. M. Chuiko, A generalized matrix differential-algebraic equation. (Russian) *Ukr. Mat. Visn.* **12** (2015), no. 1, 11–26; translation in *J. Math. Sci. (N.Y.)* **210** (2015), no. 1, 9–21. 21.
- [3] S. M. Chuiko, The Green operator of a generalized matrix differential-algebraic boundary value problem. (Russian) *Sibirsk. Mat. Zh.* **56** (2015), no. 4, 942–951; translation in *Sib. Math. J.* **56** (2015), no. 4, 752–760.
- [4] S. M. Chuiko, The generalized Green operator of a Noetherian boundary value problem for a matrix differential equation. (Russian) *Izv. Vyssh. Uchebn. Zaved. Mat.* **2016**, no. 8, 74–83; translation in *Russian Math. (Iz. VUZ)* **60** (2016), no. 8, 64–73.
- [5] S. M. Chuiko, On the regularization of a matrix differential-algebraic boundary value problem. (Russian) *Ukr. Mat. Visn.* **13** (2016), no. 1, 76–90; translation in *J. Math. Sci. (N.Y.)* **220** (2017), no. 5, 591–602.
- [6] A. M. Samoilenko, M. I. Shkil and V. P. Yakovets, *Linear Systems of Differential Equations with Degenerations*. Vyshcha Shkola, Kyiv, 2000.