Necessary and Sufficient Conditions of Disconjugacy for Fourth Order Linear Ordinary Differential Equations

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1 Introduction

In this study we consider the question of the disconjugacy on the interval $I := [a,b] \subset [0,+\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t), \tag{1.1}$$

where $p : I \to \mathbb{R}$ is a Lebesgue integrable function.

The disconjugacy results obtained in this study complete Kondrat'ev's second comparison theorem for $n = 4$, and significantly improve some other known results (see Remarks 2.1, 2.2, 2.4).

Here we use the following notations.

$\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_0^- =]-\infty,0]$, $\mathbb{R}_0^+ = [0,+\infty[$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}^3(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I;\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \quad \big(x(t) \succcurlyeq y(t)\big) \text{ for } t \in I$$

means that $x \leq y$ ($x \geq y$) and $x \neq y$. Also we use the notation $[x]_\pm = (|x| \pm x)/2$.

By a solution of equation (1.1) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$ which satisfies equation (1.1) a. e. on I.

For the formulation of our results we need the following definitions.

Definition 1.1. Equation (1.1) is said to be disconjugate (non oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.1) is oscillatory on I.

Definition 1.2. We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and equation (1.1), under the conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1), \tag{1.2}$$

has a solution u such that $u(t) > 0$ $t \in]a,b[$.

Definition 1.3. We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and equation (1.1), under the conditions

$$u(a)=0, \quad u^{(i)}(b)=0 \ \ (i=0,1,2), \tag{1.3}$$

has a solution u, such that $u(t)>0$ $t \in \,]a,b[\,$.

Remark 1.1. Let $p \in L(I;\mathbb{R}_0^+)$ ($p \in L(I;\mathbb{R}_0^-)$), and consider the equation

$$u^{(4)}(t)=\lambda^4 p(t)u(t) \ \text{ for } \ t \in I. \tag{1.4}$$

Then the set $D_+(I)$ ($D_-(I)$) can be interpreted as a set of functions $p: I \to \mathbb{R}_0^+$ ($\mathbb{R}_0^-$) for which $\lambda=1$ is the first eigenvalue of problem (1.4), (1.2) ((1.4), (1.3)).

2 Main results

2.1 Disconjugacy of equation (1.1) with non-negative coefficient

Theorem 2.1. *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation* (1.1) *is disconjugate on I iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \ \text{ for } \ t \in I. \tag{2.1}$$

Let $\lambda_1>0$ be the first eigenvalue of the problem

$$u^{(4)}(t)=\lambda^4 u(t), \quad u^{(i)}(0)=0, \quad u^{(i)}(1)=0 \ \ (i=0,1), \tag{2.2}$$

then due to Remark 1.1 we have $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$, and the following corollary is true.

Corollary 2.1. *Equation* (1.1) *is disconjugate on I if*

$$0 \le p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \text{ for } \ t \in I, \tag{2.3}$$

and is oscillatory on I if

$$p(t) \ge \frac{\lambda_1^4}{(b-a)^4} \ \text{ for } \ t \in I. \tag{2.4}$$

Remark 2.1. It is well-known that the first eigenvalue λ_1 of problem (2.2) is the first positive root of the equation $\cos\lambda \cdot \cosh\lambda = 1$, and $\lambda_1 \approx 4.73004$ (see [3]). Also in Theorem 3.1 of paper [3] it was proved that the equation $u^{(4)}=\lambda^4 u$ is disconjugate on $[0,1]$ if $0 \le \lambda < \lambda_1$.

Even if both conditions (2.3) and (2.4) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following theorem.

Theorem 2.2. *Let $p \in L(I;\mathbb{R}_0^+)$, and there exists $M \in \mathbb{R}_0^+$ such that*

$$M\,\frac{b-a}{2} + \int\limits_a^b [p(s)-M]_+\,ds \le \frac{192}{(b-a)^3}\,. \tag{2.5}$$

Then equation (1.1) *is disconjugate on I.*

2.2 Disconjugacy of equation (1.1) with non-positive coefficient

Theorem 2.3. *Let* $p \in L(I;\mathbb{R}_0^-)$. *Then equation* (1.1) *is disconjugate on* I *iff there exists* $p_* \in D_-(I)$ *such that*

$$p(t) \succcurlyeq p_*(t) \ \ for \ \ t \in I. \tag{2.6}$$

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = -\lambda^4 u(t), u^{(i)}(0) = 0 \ \ (i = 0, 1, 2), \quad u(1) = 0, \tag{2.7}$$

then due to Remark 1.1 we have $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$, and the following corollary is true.

Corollary 2.2. *Equation* (1.1) *is disconjugate on* I *if*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \le 0 \ \ for \ \ t \in I, \tag{2.8}$$

and is oscillatory on I *if*

$$p(t) \le -\frac{\lambda_2^4}{(b-a)^4} \ \ for \ \ t \in I. \tag{2.9}$$

Remark 2.2. In Theorem 4.1 of [3] the following is proved: Let λ_2 be the first positive root of the equation $\tanh \frac{\lambda}{\sqrt{2}} = \tan \frac{\lambda}{\sqrt{2}}$ ($\lambda_2 \approx 5.553$). Then the equation $u^{(4)} = -\lambda^4 u$ is disconjugate on $[0, 1]$ if $0 \le \lambda < \lambda_2$.

Even if both conditions (2.8) and (2.9) are violated, the question on the disconjugacy of equation (1.1) can be answered by the following

Theorem 2.4. *Let* $p \in L(I;\mathbb{R}_0^-)$ *be such that there exists* $M \in \mathbb{R}_0^+$ *with*

$$M\frac{495}{1024}(b-a) + \int_a^b [p(s) + M]_- \, ds \le \frac{110}{(b-a)^3}\,. \tag{2.10}$$

Then equation (1.1) *is disconjugate on* I.

2.3 Disconjugacy of equation (1.1) with not necessarily constant sign coefficient

Theorem 2.5. *Let* $p_* \in D_-(I)$ *and* $p^* \in D_+(I)$. *Then for an arbitrary function* $p \in L(I;\mathbb{R})$ *such that*

$$p_*(t) \preccurlyeq -[p(t)]_-, \quad [p(t)]_+ \preccurlyeq p^*(t) \ \ for \ \ t \in I, \tag{2.11}$$

equation (1.1) *is disconjugate on* I.

The theorem is optimal in the sense that inequalities (2.11) can not be replaced by the condition $p_* \le p \le p^*$.

Remark 2.3. Let $p_1, p_2 : [a, b] \to \mathbb{R}$ be continuous functions such that the equations

$$u^{(4)}(t) = p_1(t)u(t), \quad u^{(4)}(t) = p_2(t)u(t), \tag{2.12}$$

are disconjugate on I, then due to Kondrat'ev's second comparison theorem, if $p_1 \le p \le p_2$, then equation (1.1) is disconjugate too. Here coefficients p_1 and p_2 should not necessarily be constant sign functions, while in Theorem 2.5 for the permissible coefficients p_1 and p_2, equations (2.12) should not necessarily be disconjugate and continuous. For this reason, if

$$p(t) = \lambda_1^4 \Big[\cos \frac{2\pi t}{n}\Big]_+ - \lambda_2^4 \Big[\cos \frac{2\pi t}{n}\Big]_-,$$

then from Theorem 2.5 it follows the disconjugacy of equation (1.1) on $[0,1]$ for all $n \in N$ (see Corollary 2.4), while this fact does not follow from Kondrat'ev's theorem.

Corollary 2.3. *Let $p_* \in D_-(I)$, $p^* \in D_+(I)$, and*

$$\operatorname{mes}\Big\{t \in I \mid \ p_*(t) \cdot p^*(t) \neq 0\Big\} > 0.$$

Then equation (1.1) *with $p = p_* + p^*$ is disconjugate on I.*

From Theorem 2.5 with

$$p_* := -\frac{\lambda_2^4}{(b-a)^4} \ \text{ and } \ p^* := \frac{\lambda_1^4}{(b-a)^4}$$

we obtain

Corollary 2.4. *et $\lambda_1 > 0$ and $\lambda_2 > 0$ be the first eigenvalues of problems* (2.2) *and* (2.7)*, respectively, and the function $p \in L(I;\mathbb{R})$ admits the inequalities*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \text{ for } \ t \in I.$$

Then equation (1.1) *is disconjugate on I.*

Remark 2.4. If we take into account that $\lambda_1^4 \approx 501$ and $\lambda_2^4 \approx 951$, then it is clear that Corollary 2.4 significantly improves Coppel's well-known condition

$$\max_{t \in [a,b]} |p(t)| \leq \frac{128}{(b-a)^4}\,,$$

proved in [1], which for $p \in C(I;\mathbb{R})$ guarantees the disconjugacy of equation (1.1) on I.

References

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