On Solvability of Inhomogeneous Boundary-Value Problems in Fractional Sobolev Spaces

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1 Introduction

Boundary-value problems for systems of ordinary differential equations arise in many problems of analysis and its applications. Unlike Cauchy problems, the solutions to such problems may not exist or may not be unique. Thus, it is interesting to investigate the nature of the solvability of inhomogeneous boundary-value problems in the functional Sobolev and Sobolev–Slobodetskiy spaces and the dependence of their solutions on the parameter. For Fredholm boundary-value problems, similar issues have been investigated in papers [1, 2, 4, 5, 7]. The case of underdefined or overdefined boundary-value problems in Sobolev spaces was investigated in paper [3].

2 Statement of the problem

Let a finite interval $(a,b)\subset\mathbb{R}$ and parameters

$$\{m,l\}\subset\mathbb{N},\ \ s\in(1,\infty)\setminus\mathbb{N},\ \ 1\le p<\infty$$

be given. By $W_p^n:=W_p^n([a,b];\mathbb{C})$ we denote a complex Sobolev space and set $W_p^0:=L_p$. By

$$(W_p^n)^m:=W_p^n([a,b];\mathbb{C}^m)\ \text{ and }\ (W_p^n)^{m\times m}:=W_p^n([a,b];\mathbb{C}^{m\times m})$$

we denote the Sobolev spaces of vector functions and matrix functions, respectively, with elements from the function space W_p^n. By $\|\cdot\|_{n,p}$ we denote the norms in these spaces. They are defined as the sums of the corresponding norms of all elements of a vector-valued or matrix-valued function in W_p^n. The space of functions (scalar functions, vector functions, or matrix functions) in which the norm is introduced is always clear from the context. For $m=1$ all these spaces coincide. It is known that W_p^n are separable Banach spaces.

We denote by $W_p^s:=W_p^s([a,b];\mathbb{C})$ where $1\le p<\infty$ and $s>1$, is not integer, the Sobolev–Slobodetskiy space of all complex-valued functions belonging to Sobolev space $W_p^{[s]}$ and satisfying the condition

$$\|f\|_{s,p}:=\|f\|_{[s],p}+\bigg(\int\limits_a^b\int\limits_a^b\frac{|f^{[s]}(x)-f^{[s]}(y)|^p}{|x-y|^{1+\{s\}p}}\,dx\,dy\bigg)^{1/p}<+\infty,$$

where $[s]$ is the integer part, and $\{s\}$ is the fractional part of the number s. Here, we recall that $\|\cdot\|_{[s],p}$ is the norm in the Sobolev space $W_p^{[s]}$. This equality defines the norm $\|f\|_{s,p}$ in the space W_p^s.

Consider a linear boundary-value problem on a finite interval (a,b) for the system of m first-order scalar differential equations

$$(Ly)(t) := y'(t) + A(t)y(t) = f(t), \ \ t \in (a,b), \tag{2.1}$$

$$By = c, \tag{2.2}$$

where the matrix function $A(\cdot)$ belongs to the space $(W_p^s)^{m\times m}$, the vector function $f(\cdot)$ belongs to the space $(W_p^s)^m$, the vector c belongs to the space $\mathbb{C}^l$, and B is a linear continuous operator

$$B : (W_p^{s+1})^m \to \mathbb{C}^l. \tag{2.3}$$

The boundary condition (2.2) consists of l scalar boundary conditions for the system of m differential equations of the first order. We represent vectors and vector functions in the form of columns. A solution to the boundary-value problem (2.1), (2.2) is understood as a vector function $y \in (W_p^{s+1})^m$, satisfying equation (2.1) for $s > 1 + 1/p$ everywhere and, for $s \le 1 + 1/p$, almost everywhere on (a,b) and equality (2.2) specifying l scalar boundary conditions. The solutions to equation (2.1) fill the space $(W_p^{s+1})^m$, if its right-hand side $f(\cdot)$ runs through the space $(W_p^s)^m$. Hence, the boundary condition (2.2) is the most general condition for this equation and includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multipoint problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives of integer or fractional order k of required vector–functions, where $0 < k < s + 1$.

The main purpose of this work is to establish whether the boundary-value problem (2.1), (2.2) has the Fredholm property; to find its index and the dimension of the cokernel and the kernel of the operator of an inhomogeneous boundary-value problem in terms of the properties of a special rectangular numerical matrix and to investigate its stability. In the case of Sobolev spaces of integer order, similar results were obtained in [6].

3 Main results

We rewrite the inhomogeneous boundary-value problem (2.1), (2.2) in the form of a linear operator equation

$$(L,B)y = (f,c),$$

where (L,B) is a linear operator in the pair of Banach spaces

$$(L,B) : (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l. \tag{3.1}$$

Let X and Y be Banach spaces. Recall that a linear continuous operator $T : X \to Y$ is called a Fredholm operator, if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If the operator is a Fredholm one, then its range $T(X)$ is closed in Y, and the index

$$\operatorname{ind} T := \dim\ker T - \dim \frac{Y}{T(X)}$$

is finite (see, e.g., [6, Lemma 19.1.1]).

Theorem 3.1. *The linear operator* (3.1) *is a bounded Fredholm operator with index* $m - l$.

Denote by $Y(\,\cdot\,) \in (W_p^s)^{m\times m}$ the unique solution to a linear homogeneous matrix equation

$$Y'(t) + A(t)Y(t) = O_m, \ \ t \in (a,b) \tag{3.2}$$

with the initial condition

$$Y(a) = I_m. \tag{3.3}$$

Here, O_m are zero matrices, and I_m are identity $(m\times m)$ matrices. The unique solution to the Cauchy problem (3.2), (3.3) belongs to the space $(W_p^{s+1})^{m\times m}$.

By $[BY]$ we denote a numerical matrix of dimension $(m\times l)$ whose i-th column is a result of the action of the operator B from (2.3) on i-th column of the matrix function $Y(\,\cdot\,)$, $i\in\{1,\dots,m\}$.

Definition. A rectangular numerical matrix

$$M(L,B) = [BY] \in \mathbb{C}^{m\times l} \tag{3.4}$$

is called the characteristic matrix for the inhomogeneous boundary-value problem (2.1), (2.2).

Here, m is the number of scalar differential equations of system (2.1), and l is the number of scalar boundary conditions.

Theorem 3.2. *The dimensions of the kernel and cokernel of operator* (3.1) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix* (3.4)*, respectively:*

$$\dim\ker(L,B) = \dim\ker(M(L,B)),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}(M(L,B)).$$

A criterion for the invertibility of the operator (L,B) follows from Theorem 3.2, i.e., the condition under which problem (2.1), (2.2) possesses a unique solution, and this solution continuously depends on the right-hand sides of the differential equation and the boundary condition.

Corollary 3.1. *Operator (L,B) is invertible if and only if $l=m$ and the square matrix $M(L,B)$ is nondegenerate.*

4 Application

In addition to problem (2.1), (2.2) we consider the sequence of inhomogeneous boundary-value problems

$$L(k)y(t,k) := y'(t,k) + A(t,k)y(t,k) = f(t,k), \ \ t\in(a,b), \tag{4.1}$$

$$B(k)y(\,\cdot\,,k) = c(k), \ \ k\in\mathbb{N}, \tag{4.2}$$

where the matrix functions $A(\,\cdot\,,k)$, the vector functions $f(\,\cdot\,,k)$, the vectors $c(k)$ and linear continuous operators $B(k)$ satisfy the above conditions for problem (2.1), (2.2).

With the boundary-value problem (4.1), (4.2) we associate a sequence of linear continuous operators

$$(L(k),B(k)) : (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l$$

and a sequence of characteristic matrices

$$M(L(k),B(k)) = [B(k)Y(\,\cdot\,,k)] \subset \mathbb{C}^{m\times l},$$

depending on the parameter $k\in\mathbb{N}$.

We now formulate a sufficient condition for the convergence of the characteristic matrices $M(L(k),B(k))$ to the matrix $M(L,B)$.

Theorem 4.1. *If the sequence of operators $(L(k), B(k))$ converges strongly to the operator (L, B), for $k \to \infty$, then the sequence of characteristic matrices $M(L(k), B(k))$ converges to the matrix $M(L, B)$.*

Corollary 4.1. *Under the assumptions from Theorem 4.1, the following inequalities hold*

$$\dim\ker(L(k), B(k)) \leq \dim\ker(L, B),$$
$$\dim\operatorname{coker}(L(k), B(k)) \leq \dim\operatorname{coker}(L, B)$$

for sufficiently large k.

In particular:

1. If $l = m$ and the operator(L, B) is invertible, then the operators $(L(k), B(k))$ are also invertible for large k;
2. If the boundary-value problem (2.1), (2.2) has a solution for any values of the right-hand sides, then the boundary-value problems (4.1), (4.2) also have a solution for large k;
3. If the boundary-value problem (2.1), (2.2) has a unique solution, then problems (4.1), (4.2) also have a unique solution for each sufficiently large k.

References

[1] O. M. Atlasiuk and V. A. Mikhailets, Fredholm one-dimensional boundary-value problems in Sobolev spaces. (Ukrainian) *Ukr. Mat. Zh.* **70** (2018), no. 10, 1324–1333; translation in *Ukr. Math. J.* **70** (2019), no. 10, 1526–1537.

[2] O. M. Atlasiuk and V. A. Mikhailets, Fredholm one-dimensional boundary-value problems with parameters in Sobolev spaces. (Ukrainian) *Ukr. Mat. Zh.* **70** (2018), no. 11, 1457–1465; translation in *Ukr. Math. J.* **70** (2019), no. 11, 1677–1687.

[3] O. M. Atlasiuk and V. A. Mikhailets, On solvability of inhomogeneous boundary-value problems in Sobolev spaces. *Dopov. Nats. Acad. Nauk Ukr. Mat. Pridozn. Tekh. Nauki* **2019**, no. 11, 3–7.

[4] E. V. Gnyp, T. I. Kodlyuk and V. A. Mikhailets, Fredholm boundary-value problems with parameter in Sobolev spaces. *Ukraïn. Mat. Zh.* **67** (2015), no. 5, 584–591; *Ukrainian Math. J.* **67** (2015), no. 5, 658–667.

[5] Y. Hnyp, V. Mikhailets and A. Murach, Parameter-dependent one-dimensional boundary-value problems in Sobolev spaces. Electron. *J. Differential Equations* **2017**, Paper No. 81, 13 pp.

[6] L. Hörmander, *The Analysis of Linear Partial Differential Operators.* III. *Pseudodifferential Operators.* Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 274. Springer-Verlag, Berlin, 1985.

[7] T. I. Kodliuk and V. A. Mikhailets, Solutions of one-dimensional boundary-value problems with a parameter in Sobolev spaces. (Russian) *Ukr. Mat. Visn.* **9** (2012), no. 4, 546–559; translation in *J. Math. Sci., New York* **190** (2013), 589–599.