# Disconjugacy and Green's Functions Sign for Some Two-Point Boundary Value Problems for Fourth Order Ordinary Differential Equations

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## 1 Introduction

Here we consider the question of the disconjugacy on the interval $I := [a,b] \subset [0,+\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t) - \mu u(t) \ \text{ for } \ t \in I, \tag{1.1}$$

when $p : I \to \mathbb{R}$ is Lebesgue integrable function, $\mu \in \mathbb{R}$, and the question of the Green's functions sign for equation (1.1) under one of the following two-point boundary conditions

$$u(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1,2), \tag{1.2$_1$}$$

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1), \tag{1.2$_2$}$$

$$u^{(i)}(a) = 0 \ \ (i = 0,1,2), \quad u(b) = 0. \tag{1.2$_3$}$$

There are established the optimal sufficient conditions of disconjugacy of equation (1.1) when the coefficient $p$ is not necessarily constant sign function. On the basis of these results we prove the necessary and sufficient conditions of non-negativity (non-positivity) of Green's function for problems (1.1), (1.2$_\ell$) ($\ell \in \{1,2,3\}$), which are formulated in a terminology of eigenvalues of problems under the consideration.

Here we use the following notations.

$\mathbb{R} = ]-\infty, +\infty[\,$, $\mathbb{R}_0^- = ]-\infty, \, 0]\,$, $\mathbb{R}_0^+ = [0, +\infty[\,$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : \ t \in I\}$.

$\widetilde{C}^3(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I;\,\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \quad \big(x(t) \succcurlyeq y(t)\big) \ \text{ for } \ t \in I,$$

means that $x \le y$ ($x \ge y$) and $x \ne y$.

Also we use the notation $[x]_\pm = \frac{|x| \pm x}{2}$ .

By a solution of equation (1.1) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$ which satisfies equation (1.1) a.e. on $I$.

Also we need the following definition.

**Definition 1.1.** Equation

$$u^{(4)}(t) = p(t)u(t) \tag{1.3}$$

is said to be disconjugate (non-oscillatory) on $I$, if every nontrivial solution $u$ has less then four zeros on $I$, the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.3) is oscillatory on $I$.

The given study is based on our previous results from the paper [3] and the results of A. Cabada and R. Enguica from [1]. For the formulation of our main results we need the following definitions and propositions from our previous paper.

**Definition 1.2.** We will say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and problem (1.3), $(1.2_2)$ has a solution $u$ such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,.$$

**Definition 1.3.** We will say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and problem (1.3), $(1.2_3)$ has a solution $u$ such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,\, b[\,.$$

The propositions below are Theorems 2, 4, and 6, respectively, from the paper [3].

**Proposition 1.1.** *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation* (1.3) *is disconjugate on $I$ iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \ \text{ on } I.$$

**Proposition 1.2.** *Let $p \in L(I;\mathbb{R}_0^-)$. Then equation* (1.3) *is disconjugate on $I$ iff there exists $p_* \in D_-(I)$ such that*

$$p(t) \succcurlyeq p_*(t) \ \text{ on } I.$$

**Proposition 1.3.** *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\ \mathbb{R})$ such that*

$$p_*(t) \preccurlyeq -[p(t)]_- \,, \quad [p(t)]_+ \preccurlyeq p^*(t) \ \text{ on } I, \tag{1.4}$$

*equation* (1.3) *is disconjugate on $I$.*

**Remark 1.1.** From Proposition 1.1 (Proposition 1.2) it is clear that the structure of the set $D_+(I)$ $(D_-(I))$ is such that if $x, y \in D_+(I)(x, y \in D_-(I))$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ hold.

**Remark 1.2.** If $\lambda_1$ $(\lambda_2)$ is the first positive eigenvalue of the problem

$$\begin{gathered} u^{(4)}(t) = \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \ \ (i = 0, 1) \\ (u^{(4)}(t) = -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \ \ (i = 0, 1, 2), \quad u(1) = 0), \end{gathered}$$

then

$$\frac{\lambda_1^4}{(b-a)^4} \in D_+(I) \quad \Big( -\frac{\lambda_2^4}{(b-a)^4} \in D_-(I) \Big).$$

Also it is well-known (see [1] or [2]), that $\lambda_1 \approx 4.73004$ and $\lambda_2 \approx 5.553$.

## 2 Main results

First we consider the results on disconjugacy of equation (1.1) on the interval $[a,b]$.

**Theorem 2.1.** *Let $p \in L(I;\mathbb{R})$, conditions*

$$\begin{aligned} \alpha_p &:= \inf_{p^* \in D_+(I)} \Big\{ \operatorname{ess\,sup}_{t\in I} \big\{ p(t) - p^*(t) \big\} \Big\} \leq 0, \\ \beta_p &:= \sup_{p_* \in D_-(I)} \Big\{ \operatorname{ess\,inf}_{t\in I} \big\{ p(t) - p_*(t) \big\} \Big\} \geq 0, \end{aligned} \tag{2.1}$$

*hold, and*

$$\alpha_p \neq \beta_p.$$

*Then equation* (1.1) *is disconjugate on $I$ if $\mu \in \,]\alpha_p, \beta_p[\,$.*

**Remark 2.1.** Theorem 2.1 is optimal in the sense that there exists $p \in L(I;\mathbb{R})$ such that if $\mu = \alpha_p$ or $\mu = \beta_p$, then equation (1.1) is oscillatory on $I$. Indeed, let $p(t) \equiv \frac{\lambda_1^4 - \lambda_2^4}{2(b-a)^4}$, where due to Remark 1.2 we have $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$ and $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$. Then from Remark 1.1 it immediately follows that

$$\alpha_p = p(t) - \frac{\lambda_1^4}{(b-a)^4} \ \text{ and } \ \beta_p = p(t) + \frac{\lambda_2^4}{(b-a)^4}\,.$$

Therefore if $\mu = \alpha_p$ or $\mu = \beta_p$, then equation (1.1) is oscillatory on $I$.

**Corollary 2.1.** *Let $p \in D_+(I)$. Then $\beta_p > \frac{\lambda_2^4}{(b-a)^4}$, and equation* (1.1) *is disconjugate on $I$ if $\mu \in \,]0, \beta_p[\,$.*

**Corollary 2.2.** *Let $p \in D_-(I)$. Then $\alpha_p < -\frac{\lambda_1^4}{(b-a)^4}$, and equation* (1.1) *is disconjugate on $I$ if $\mu \in \,]\alpha_p, 0[\,$.*

From the last two corollaries we immediately have

**Corollary 2.3.** *Let $\mu \in \,]0, \frac{\lambda_1^4}{(b-a)^4}]$ $\big(\mu \in [-\frac{\lambda_2^4}{(b-a)^4}, 0[\,\big)$. Then equation* (1.1) *is disconjugate on $I$ for an arbitrary $p \in D_+(I)$ $(p \in D_-(I))$.*

**Remark 2.2.** Corollaries 2.1 and 2.2 are optimal.

As it is well-known the disconjugacy is only a sufficient condition in order to ensure the constant sign of Green's function of problems $(1.3), (1.2_\ell)$ $(\ell \in \{1,2,3\})$. For this reason we introduce here theorems with necessary and sufficient conditions which guarantee that Green's function of problems $(1.1), (1.2_2)$ or $(1.1), (1.2_3)$ will be the constant sign function. Also we will find such coefficients $p$, and such values of the parameter $\mu$, for which Green's functions of the mentioned problems are constant sign functions but equation (1.1) is oscillatory on $I$ (see Remark 2.3).

**Theorem 2.2.** *Let $p \in D_+(I) \cap C(I;\mathbb{R})$. Then:*

(a) *Green's function of problem $(1.1), (1.2_2)$ is non-negative on $I \times I$ iff $\mu \in \,]0, \mu_p]$, where $\mu_p := \min\{\mu_1^*, \mu_3^*\}$, $\mu_\ell^*$ $(\ell = 1,3)$ is the first positive eigenvalue of problem $(1.1), (1.2_\ell)$;*

(b) *The estimation $\mu_p \geq \beta_p > \frac{\lambda_2^4}{(b-a)^4}$ is valid.*

**Theorem 2.3.** *Let $p \in D_-(I) \cap C(I;\mathbb{R})$. Then:*

(a) *Green's function of problem* $(1.1),(1.2_2)$ *is non-negative on* $I\times I$ *iff* $\mu\in\,]\mu_p,0]$, *where* $\mu_p$ *is the biggest negative eigenvalue of problem* $(1.1),(1.2_2)$*;*

(b) *The estimation* $\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}$ *is valid.*

**Theorem 2.4.** *Let* $p\in D_+(I)\cap C(I;\mathbb{R})$*. Then:*

(a) *Green's function of problem* $(1.1),(1.2_1)$ *is non-positive on* $I\times I$ *iff* $\mu\in[0,\mu_p[\,$, *where* $\mu_p$ *is the first positive eigenvalue of problem* $(1.1),(1.2_1)$*;*

(b) *The estimation* $\mu_p\ge\beta_p>\frac{\lambda_2^4}{(b-a)^4}$ *is valid.*

**Theorem 2.5.** *Let* $p\in D_-(I)\cap C(I;\mathbb{R})$*. Then:*

(a) *Green's function of problem* $(1.1),(1.2_1)$ *is non-positive on* $I\times I$ *iff* $\mu\in[\mu_p,0[\,$, *where* $\mu_p$ *is the biggest negative eigenvalue of problem* $(1.1),(1.2_2)$*;*

(b) *The estimation* $\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}$ *is valid.*

**Remark 2.3.** In Theorems 2.2 and 2.4 (Theorems 2.3 and 2.5) from the definition of the number $\mu_p$ it is clear that equation (1.1) is oscillatory on $I$ if $\mu=\mu_p$. Therefore from Corollary 2.1 (Corollary 2.2) it immediately follows that

$$\mu_p\ge\beta_p>\frac{\lambda_2^4}{(b-a)^4}\quad\Big(\mu_p\le\alpha_p<-\frac{\lambda_1^4}{(b-a)^4}\Big).$$

## References


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