On an Interpolation Boundary Value Problem

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1 Introduction

In the general case, the multipoint boundary value problem for a differential system

$$\mathcal{L}x = f \tag{1.1}$$

considered on the segment $[0, T]$ is a problem with boundary conditions $\ell x = \beta$ of the form

$$\ell x \equiv \sum_{i=0}^{m} \Lambda_i x(t_i) = \beta,$$

where $\{t_i\}$, $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = T$, is a fixed collection of points from $[0, T]$, Λ_i, $i = 0, \dots, m$, are given $(N \times n)$-matrices, $\beta \in R^N$. Here we consider a special case of boundary conditions that correspond to the interpolation problem as a problem of trajectories taking prescribed values at the given points:

$$x(t_i) = \alpha_i, \;\; i = 0, \dots, m. \tag{1.2}$$

The problem under consideration can be interpreted as a part of the routing problem (see, for instance, [2]) namely as the task of implementing the route. Similar problems arise in Economic Dynamics [4,6] where α_i, $i = 1, \dots, m$, are given values of indicators to a modeled economic system at the time moments t_i.

In the case with no constraints with respect to the right-hand side f, for any collection α_i, $i = 0, \dots, m$, there exists $f : [0, T] \to R^n$ that provides the solvability of (1.1), (1.2). In contrast to this, if f is constrained by the inequalities

$$a_i \leqslant f_i(t) \leqslant b_i, \;\; i = 1, \dots, n, \;\; t \in [0, T], \tag{1.3}$$

there arises the task to describe a set of α_i, $i = 0, \dots, m$, for which (1.1), (1.2) is solvable.

Below we propose a way of constructing a hypercube P_N in R^N, $N = mn$ such that the condition $\alpha = \operatorname{col}(\alpha_1, \dots, \alpha_m) \in P_N$ is a sufficient condition to the solvability of (1.1), (1.2) in the sense that there exists an f with (1.3) such that the corresponding trajectory takes the values prescribed by (1.2).

First we descript a class of functional differential equations with linear Volterra operators and appropriate spaces where those are considered. The main relationships that allow to obtain sufficient conditions of the solvability to the problem under consideration are proposed. An illustrative example of application of the main theorem is presented.

2 Main constructions

We consider a quite broad class of functional differential systems with aftereffect and follow the notation and basic statements of the general functional differential theory in the part concerning linear systems with aftereffect [1,3].

Let $L^n = L^n[0,T]$ be the Lebesgue space of all summable functions $z : [0,T] \to R^n$ defined on a finite segment $[0,T]$ with the norm

$$\|z\|_{L^n} = \int\limits_0^T |z(t)|\,dt,$$

where $|\cdot|$ is a norm in R^n. Denote by $AC^n = AC^n[0,T]$ the space of absolutely continuous functions $x : [0;T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In what follows we will use some results from [1,3].

We consider the case of the system (1.1) with a linear bounded Volterra operator $\mathcal{L} : AC^n \to L^n$ such that the general solution of the equation (1.1) has the form

$$x(t) = X(t)x(0) + \int\limits_0^t C(t,s)f(s)\,ds, \tag{2.1}$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$, $C(t,s)$ is the Cauchy matrix. A broad class of operators $\mathcal{L}$ with the property (2.1) is described, for instance, in [5].

The properties of the Cauchy matrix used below are studied in detail in [3]. Without loss of generality we put in the sequel $x(0) = \alpha_0 = 0$ and $a_i < 0$, $b_i > 0$, $i = 1,\dots,n$.

Denote by E_n the identity $(n \times n)$-matrix, e_n^i stands for the i-th (from above) row of E_n. We define

$$P_0 = \prod_{i=1,\dots,n}^{n} [a_i, b_i].$$

Fix a positive integer K and put $\Delta = T/K$.

Let us describe the main steps and constructions on the way to sufficient solvability conditions for (1.1), (1.2) with constraints (1.3).

Define $(n \times N)$-matrix $M(s) = \operatorname{col}(M_1(s),\dots,M_m(s))$ by the equalities

$$M_i(s) = \chi_i(s)C(t_i,s), \quad i = 1,\dots,m, \tag{2.2}$$

where $\chi_i(s)$ is the characteristic function of the segment $[0,t_i]$.

For any $i = 1,\dots,N$ and $j = 1,\dots,K$ consider the following two linear programming problems

$$e_N^i M(\Delta\cdot j)v \to \max, \quad v \in P_0 \;\text{ and }\; e_N^i M(\Delta\cdot j)v \to \min, \quad v \in P_0. \tag{2.3}$$

Denote by v_{ij}^+ and v_{ij}^- solutions to the above problems, respectively,

$$v_{ij}^+ = \operatorname{argmax}\big(e_N^i M(\Delta\cdot j)v,\ v \in P_0\big) \;\text{ and }\; v_{ij}^- = \operatorname{argmin}\big(e_N^i M(\Delta\cdot j)v,\ v \in P_0\big). \tag{2.4}$$

Next define

$$d_i = e_N^i \int_0^T M(s) \sum_{j=1}^K v_{ij}^+ \chi_{[\Delta(j-1),\Delta j]}(s)\, ds, \quad i = 1, \dots, N, \tag{2.5}$$

$$c_i = e_N^i \int_0^T M(s) \sum_{j=1}^K v_{ij}^- \chi_{[\Delta(j-1),\Delta j]}(s)\, ds, \quad i = 1, \dots, N, \tag{2.6}$$

$$\mathcal{D}_k = \operatorname{diag}(d_1, \dots, d_k), \ \ \mathcal{C}_k = \operatorname{diag}(c_1, \dots, c_k), \ \ k = 1, \dots, N. \tag{2.7}$$

Introduce the following matrices:

$$\begin{gathered} A = \begin{pmatrix} \mathcal{D}_{N-1} & 0 \\ \mathcal{C}_{N-1} & 0 \\ \mathcal{D}_{N-1} & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} F_{N-1}^i \\ d_N e_N^N \end{pmatrix}, \ \ i = 1, \dots, 2(N-1), \\ B_i = \begin{pmatrix} F_{N-1}^i \\ c_N e_N^N \end{pmatrix}, \ \ i = 2(N-1)+1, \dots, 4(N-1). \end{gathered} \tag{2.8}$$

Here F_{N-1}^i is the i-th group of $N-1$ consecutive rows of matrix A. Each B_i gives a collection of N points in R^N that define the corresponding hyperplane and the corresponding halfspace with the zero point. The intersection of all above hyperplanes is the polyhedron of all attainable points in the considered interpolation problem. Now our task is to construct a hypercube that is a subset of the polyhedron.

For each of $4(N-1)$ mentioned hyperplanes we define the distance ρ_k from the hyperplane to the origin. Namely, let

$$p_1^k z_1 + p_2^k z_2 + \dots + p_N^k z_n + q^k = 0$$

be the equation of the k-th hyperplane. Then we have

$$\rho_k = \frac{|q_k|}{\sqrt{\sum\limits_{i=1}^N (p_i^k)^2}}\,, \quad k = 1, \dots, 4(N-1). \tag{2.9}$$

It is clear that the ball $S(0,\rho)$ with the radius $\rho = \min(\rho_k,\ k = 1, \dots, 4(N-1))$ centered by the origin is a subset of the polyhedron defined by the all above hyperplanes, and, for any $\alpha = \operatorname{col}(\alpha_1, \dots, \alpha_m) \in S(0,\rho)$, the problem $(1.1), (1.2), (1.3)$ is solvable. Finally we define the cube

$$P_N = \Big\{ z \in R^N : \ \max |z_i| \leqslant \frac{\rho}{\sqrt{N}}\,, \ i = 1, \dots, N \Big\}.$$

Thus we obtain

Theorem. *Let the set P_N be defined by the relationships* (2.2)–(2.9). *Then the interpolation problem* $(1.1), (1.2), (1.3)$ *is solvable for any $\alpha \in P_N$.*

3 An example

Following [5], consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t-1) + f_1(t), \\ \dot{x}_2(t) &= -x_2(t) + f_2(t); \end{aligned} \quad t \in [0,3] \tag{3.1}$$

where $x_2(s) = 0$ if $s < 0$, with the initial conditions

$$x_1(0) = 0, \quad x_2(0) = 0, \tag{3.2}$$

and the boundary conditions

$$x_1(2) = \alpha_1, \;\; x_2(2) = \alpha_2, \;\; x_1(3) = \alpha_3, \;\; x_2(3) = \alpha_4. \tag{3.3}$$

The right-hand sides $f_1(t)$ and $f_2(t)$ are constrained by the inequalities

$$-0.1 \leqslant f_1(t) \leqslant 0.4, \;\; -0.2 \leqslant f_2(t) \leqslant 0.5. \tag{3.4}$$

Here we have

$$C(t,s) = \begin{pmatrix} 1 & \int\limits_s^t \chi_{[1,3]}(\tau)\chi_{[0,\tau-1]}(s)\exp(1-\tau+s)\,d\tau \\ 0 & \exp(s-t) \end{pmatrix},$$

After calculation the integral in C_{12} for $t = 2$ and $t = 3$ we obtain

$$C(2,s) = \begin{pmatrix} 1 & \begin{cases} 1 - e^{s-1}, & s \in [0,1] \\ 0 & \text{otherwise} \end{cases} \\ 0 & \exp(s-2) \end{pmatrix}, \quad C(3,s) = \begin{pmatrix} 1 & \begin{cases} 1 - e^{s-2}, & s \in [0,2] \\ 0 & \text{otherwise} \end{cases} \\ 0 & \exp(s-3) \end{pmatrix},$$

The elements $M_{ij}(s)$ of $M(s)$ are defined by the equalities

$$M_{11}(s) = \chi_{[0,2]}(s), \;\; M_{12}(s) = \chi_{[0,1]}(s)(1 - e^{(s-1)}), \quad M_{21}(s) = 0, \quad M_{22}(s) = \chi_{[0,2]}(s)e^{(s-2)},$$
$$M_{31}(s) = 1, \quad M_{32}(s) = \chi_{[0,2]}(s)(1 - e^{(s-2)}), \quad M_{41}(s) = 0, \quad M_{42}(s) = e^{(s-3)}.$$

For the case of $K = 20$, calculations by the rules (2.4), (2.5), (2.6) bring the values

$$d_1 = 0.5116, \;\; d_2 = 1.7360, \;\; d_3 = 0.4393, \;\; d_4 = 0.9409,$$
$$c_1 = -0.2046, \;\; c_2 = -0.5143, \;\; c3 = -0.1757, \;\; c_4 = -0.2594.$$

Here and below all real values are displayed to four places of decimals.

For all of 12 hyperplanes the corresponding distances $\rho_1, \dots, \rho_{12}$ are as follows:

$$\rho_1 = 0.3091, \;\; \rho_2 = 0.1809, \;\; \rho_3 = 0.1715, \;\; \rho_4 = 0.1278, \;\; \rho_5 = 0.1559, \;\; \rho_6 = 0.1629,$$
$$\rho_7 = 0.2033, \;\; \rho_8 = 0.1503, \;\; \rho_9 = 0.1447, \;\; \rho_{10} = 0.1155, \;\; \rho_{11} = 0.1350, \;\; \rho_{12} = 0.1394.$$

Thus $\rho = 0.1155$, and the inequality $\max\{\alpha_1, \dots, \alpha_4\} \leqslant 0.0577$ provides the solvability of (3.1)–(3.4).

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