On k-Point Approximations for the Izobov Sigma-Exponent

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Consider a linear differential system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A such that

$$\|A(t)\| \leq M < +\infty \ \text{ for all } \ t \geq 0.$$

We denote the Cauchy matrix of (1) by X_A and the highest Lyapunov exponent of (1) by $\lambda_n(A)$. Together with system (1) consider a perturbed system

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

with piecewise continuous and bounded perturbation matrix Q such that

$$\|Q(t)\| \leq N_Q \exp(-\sigma t), \ \ t \geq 0. \tag{3}$$

Denote the higher exponent of (2) by $\lambda_n(A+Q)$.

Let $\mathfrak{M}_\sigma(A)$ be the set of all perturbations Q satisfying condition (3) and having the appropriate dimensions. Any $Q \in \mathfrak{M}_\sigma$ is said to be a sigma-perturbation and the number

$$\nabla_\sigma(A) := \sup\big\{\lambda_n(A+Q) : \ Q \in \mathfrak{M}_\sigma(A)\big\}$$

is called [4], [6, p. 225], [5, p. 214] the highest sigma-exponent or the Izobov exponent of system (1). It was proved in [4] that the Izobov exponent can be evaluated by means of the following algorithm:

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}} \frac{\xi_m(\sigma)}{m}$$
$$\xi_m(\sigma) = \max_{i<m} \big(\ln\|X_A(m,i)\| + \xi_i(\sigma) - \sigma i\big), \quad \xi_1 = 0, \ \ i \in \mathbb{N}.$$

It was proved in [1, 7] that $\nabla_\sigma(A)$ is a convex monotonically decreasing function on $[0, +\infty[$ such that

$$\nabla_\sigma(A) = \lambda_n(A) \ \text{ for all } \ \sigma > \sigma_0(A),$$

where $2M \geq \sigma_0(A) \geq 0$ is a critical value of σ for system (1).

Some alternative representation for $\nabla_\sigma(A)$ was given in [10]. Let $\mathcal{D}(m)$ be the set of all nonempty $d \subset \{1, \dots, m-1\} \subset \mathbb{N}$. Further we assume that for each $d \in \mathcal{D}(m)$ the elements of d are arranged in the increasing order, so that $d_1 < d_2 < \dots < d_s$ and $d = \{d_1, d_2, \dots, d_s\}$, where $s = |d|$ is the number of elements of the set d. We also put

$$\|d\| := d_1 + \dots + d_s \ \text{ for } \ d \in \mathcal{D}(m)$$

and
$$\|d\| := 0 \ \text{ for } \ d = \varnothing.$$
In addition, for the sake of convenience we assume that $d_0 = 0$ and $d_{s+1} = m$ for each $d \in \mathcal{D}_0(m) := \mathcal{D}(m) \cup \{\varnothing\}$. Note that we do not include these additional elements into the set d. Under the above assumptions, let us define the quantity $\Xi(m, d)$ as
$$\Xi(m, d) := \sum_{i=0}^{s} \ln \|X_A(d_{i+1}, d_i)\|,$$
where $m \in \mathbb{N}$, $d \in \mathcal{D}(m)$ and $s := |d|$. From [2, 10] we can assert that
$$\xi_m(\sigma) = \max_{d \in \mathcal{D}_0(m)} \big(\Xi(m, d) - \sigma\|d\|\big).$$
Thus, we have
$$\nabla_\sigma(A) = \overline{\lim_{m \to \infty}}\, m^{-1} \max_{d \in \mathcal{D}_0(m)} \big(\Xi(m, d) - \sigma\|d\|\big). \tag{4}$$

The main advantage of the $\nabla_\sigma(A)$ is attainability. By virtue of this property we are sure that for each $\varepsilon > 0$ there exists a perturbed system (2) with $Q \in \mathfrak{M}_\sigma$ such that
$$\lambda_n(A + Q) > \nabla_\sigma(A) - \varepsilon.$$
When constructing perturbations Q to provide the values of $\lambda_n(A+Q)$ close to $\nabla_\sigma(A)$, we urgently need to know some (or all) sequences $d(m) \in \mathcal{D}_0(m)$, $m \in \mathbb{N}$, such that
$$\nabla_\sigma(A) = \lim_{m \to \infty} m^{-1} \big(\Xi(m, d(m)) - \sigma\|d(m)\|\big). \tag{5}$$
A primary information on this issue is given by the following statement (see Property A in [4] and Lemma 9.1 in [5, p. 215]).

Proposition 1. *If $b \in \mathcal{D}_0(m)$ satisfies the condition*
$$\xi_m(\sigma) = \Xi(m, b) - \sigma\|b\|,$$
then
$$b_{i+1} - b_i \geq \frac{\sigma}{2M}\, b_i \ \textit{ for each } \ i \in \{1, \dots, s\},$$
where $b = \{b_1, \dots, b_s\}$, $s = |b|$.

Based on the theory of characteristic vectors (see [3,8]) and some results on Malkin estimates [9] we can assert that some information on the sequences $d(m)$ in (5) can be extracted from the slopes of supporting lines to the graph of $\nabla_\sigma(A)$. Since the results of this type are not available yet, here we consider some simplified version of the problem hoping to use it later to clarify the general case. To this end, we limit the number of partition points d_i in (4) on each segment $[0, m]$ by some number $k \in \mathbb{N}$.

Let $\mathcal{D}^k(m) \subset \mathcal{D}(m)$, $k \in \mathbb{N}$, be the set of all $d \in \mathcal{D}(m)$ such that $|d| \leq k$. Let us also put $\mathcal{D}_0^k(m) := \mathcal{D}^k(m) \cup \{\varnothing\}$.

Definition 1. The number
$$\nabla_\sigma^k(A) = \overline{\lim_{m \to \infty}}\, m^{-1} \max_{d \in \mathcal{D}_0^k(m)} \big(\Xi(m, d) - \sigma\|d\|\big)$$
is said to be the k-point approximation for $\nabla_\sigma(A)$.

The introduced concept inherits some basic properties of $\nabla_\sigma(A)$.

Proposition 2. *For each $k \in \mathbb{N}$ the following statements are valid.*

(i) $\nabla_\sigma(A) \ge \nabla_\sigma^k(A) \ge \lambda_n(A)$.

(ii) $\nabla_\sigma^k(A)$ *is a convex monotonically decreasing function on* $[0, +\infty[$ *such that*

$$\nabla_\sigma(A) = \lambda_n(A) \ \text{ for all } \ \sigma > \sigma_0(A).$$

(iii) *If* $b \in \mathcal{D}_0^k(m)$ *satisfies the condition*

$$\Xi(m, b) - \sigma\|b\| = \max_{d \in \mathcal{D}_0^k(m)} \big(\Xi(m, d) - \sigma\|d\|\big), \tag{6}$$

then

$$\sigma\|b\| \le 2Mm.$$

Note that (i) is an immediate consequence of Definition 1 and (iii) is a weakened analog of Proposition 1.

For each $\sigma > 0$, let us denote the set of all $b \in \mathcal{D}_0^k(m)$ satisfying condition (6) by $\mathcal{B}_\sigma^k(m)$. Obviously, $\mathcal{B}_\sigma^k(m)$ contains more then one element in general. Now take $b \in \mathcal{B}_\sigma^k(m)$ with maximal and minimal value of $m^{-1}\|b\|$ and denote them by τ_m and β_m, respectively. Finally, put

$$\mathrm{B}_\sigma(A) = \varliminf_{m \to \infty} \beta_m, \quad \mathrm{T}_\sigma(A) = \varlimsup_{m \to \infty} \tau_m.$$

Recall that a supporting line to the graph of some convex function $f : [0, +\infty[\to \mathbb{R}$ at a point $(s, f(s))$, where $s \in [0, +\infty[$, intersects the graph of the f at $(s, f(s))$ and lies beneath the graph everywhere in the domain of the function f. If f is differentiable at $s \in [0, +\infty[$, then there exists a unique supporting line at $(s, f(s))$ and this line coincides with the tangent drawn at the same point. We denote the set of slopes of supporting lines drawn at $s \in [0, +\infty[$ to the graph of f by $\mathcal{S}_s(f)$. It can be easily seen that each $\mathcal{S}_s(f)$ is a segment of the real axis.

Theorem. *The set $\mathcal{S}_\sigma(\nabla_\sigma^k(A))$ coincides with the segment* $[\mathrm{B}_\sigma(A), \mathrm{T}_\sigma(A)]$.

A similar statement is supposed to be valid for the original sigma-exponent. However, it has not yet been possible to find the reasonable representation for bounds of $\mathcal{S}_\sigma(\nabla_\sigma(A))$.

References

[1] E. A. Barabanov, Properties of the highest σ-exponent. (Russian) *Differentsial'nye Uravneniya* **18** (1982), no. 5, 739–744

[2] E. A. Barabanov, Necessary conditions for the simultaneous behavior of higher sigma-exponents of a triangular system and systems of its diagonal approximation. (Russian) *Differentsial'nye Uravneniya* **25** (1989), no. 10, 1662–1670; translation in *Differential Equations* **25** (1989), no. 10, 1146–1153 (1990).

[3] È. I. Grudo, Characteristic vectors and sets of functions of two variables and their fundamental properties. (Russian) *Differentsial'nye Uravneniya* **12** (1976), no. 12, 2115–2118.

[4] N. A. Izobov, The highest exponent of a linear system with exponential perturbations. (Russian) *Differencial'nye Uravnenija* **5** (1969), 1186–1192.

[5] N. A. Izobov, *Introduction to the Theory of Lyapunov Exponents.* BGU, Minsk, 2006.

[6] N. A. Izobov, *Lyapunov Exponents and Stability.* Stability, Oscillations and Optimization of Systems, 6. Cambridge Scientific Publishers, Cambridge, 2012.

[7] N. A. Izobov and E. A. Barabanov, The form of the highest σ-exponent of a linear system. (Russian) *Differentsial'nye Uravneniya* **19** (1983), no. 2, 359–362.

[8] E. K. Makarov, On the interrelation between characteristic functionals and weak characteristic exponents. (Russian) *Differentsial'nye Uravneniya* **30** (1994), no. 3, 393–399; translation in *Differential Equations* **30** (1994), no. 3, 362–367.

[9] E. K. Makarov, Malkin estimates for the norm of the Cauchy matrix of a linear differential system. (Russian)*Differ. Uravn.* **32** (1996), no. 3, 328–334; translation in *Differential Equations* **32** (1996), no. 3, 333–339.

[10] E. K. Makarov, I. V. Marchenko and N. V. Semerikova, On an upper bound for the higher exponent of a linear differential system with perturbations integrable on the half-axis. (Russian) D*iffer. Uravn.* **41** (2005), no. 2, 215–224, 286–287; translation in *Differ. Equ.* **41** (2005), no. 2, 227–237.