On The Behavior of Solutions to Third Order Differential Equations with General Power-Law Nonlinearities and Positive Potentia

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1 Introduction

Consider solutions to the third order differential equation with general power-law nonlinearities

$$y''' + p(x, y, y', y'')|y|^{k_0}|y'|^{k_1}\cdots|y''|^{k_2}\operatorname{sgn}(yy'y'') = 0, \tag{1.1}$$

with positive real nonlinearity exponents k_0, k_1, k_2 and positive continuous in x and Lipschitz continuous in u_0, u_1, u_2 bounded function $p(u_0, u_1, u_2)$.

The results on qualitative behavior and asymptotic estimates of positive increasing solutions for higher order nonlinear differential equations were obtained by I. T. Kiguradze and T. A. Chanturia in [9]. Questions on qualitative and asymptotic behavior of solutions to higher order Emden–Fowler differential equations ($k_1 = \cdots = k_{n-1} = 0$) were studied by I. V. Astashova in [1, 2, 5, 6].

Equation (1.1) in the case $k_0 > 0$, $k_0 \neq 1$, $k_1 = k_2 = 0$, was studied by I. Astashova in [2, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [4, 6], and proved in [3]. For third order and higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7, 8]. Qualitative properties of solutions to (1.1) in the case $p(x, y, y', y'') < 0$ were studied in [10].

2 Main results

Since solutions to equation (1.1) are not always unique, in order to obtain the full classification the following notion of μ-solutions is used.

Definition ([1]). A solution $y\colon (a, b) \to \mathbb{R}$, $-\infty \le a < b \le +\infty$ to an ordinary differential equation is a *μ-solution,* if

1. (1) the equation has no other solutions equal to y on some subinterval (a, b) and not equal to y at some point in (a, b);
2. (2) the equation either has no solution equal to y on (a, b) and defined on another interval containing (a, b) or has at least two such solutions which differ from each other at points arbitrary close to the boundary of (a, b).

Theorem 2.1. *Let the function $p(u_0, u_1, u_2)$ be continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \le p(u_0, u_1, u_2) \le M$. Then any μ-solution $y(x)$ to equation* (1.1) *according to its qualitative behavior belongs to one of the following types:*

(1) *constant function $y(x) \equiv y_0$;*

(2) *linear function* $y(x) = ax + b$, $a \neq 0$;

(3) *function with exactly one extremum.*

Remark. Let the function $p(u_0, u_1, u_2)$ be continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \leq p(x, u, v, w) \leq M$. Then the replacemenets $x \mapsto -x$ and $y(x) \mapsto -y(x)$ reduce equation (1.1) to the equation

$$z''' + \widetilde{p}(x, z, z', z'')|z|^{k_0}|z'|^{k_1}|z''|^{k_2} \operatorname{sgn}(zz'z'') = 0,$$

with the function $\widetilde{p}(u_0, u_1, u_2)$ also continuous, Lipschitz continuous in u_0, u_1, u_2 and satisfying the inequalities $0 < m \leq p(u_0, u_1, u_2) \leq M$.

Thus, it is sufficient to consider the behavior of the solutions with positive initial data near the right boundaries of their domains. In the case of a constant potential $p(u_0, u_1, u_2)$ the following results of the behavior of solutions was obtained.

Theorem 2.2. *Let* $k_2 - k_0 \neq 2$ *and* $p(u_0, u_1, u_2) \equiv p_0 > 0$. *Then any* μ*-solution* $y(x)$ *to* (1.1), *satisfying at some point* x_0 *the conditions* $y(x_0) \geq 0$, $y'(x_0) \geq 0$, $y''(x_0) > 0$ *has the following behavior near the right boundary of its domain:*

(1) *if* $0 < k_2 \leq 1$, *then there exists* $x^* < +\infty$ *such that* $y(x), y'(x) \to const$, $y''(x) \to 0$ *as* $x \to x^* - 0$;

(2) *if* $1 < k_2 \leq 2$, *then* $y(x) \to +\infty$, $y'(x) \to const$, $y''(x) \to 0$ *as* $x \to +\infty$;

(3) *if* $2 < k_2 < 2 + k_0$, *then* $y(x) \to +\infty$, $y'(x) \to const$, $y''(x) \to 0$ *as* $x \to +\infty$ *or* $y(x)$, $y'(x) \to +\infty$, $y''(x) \to 0$ *as* $x \to +\infty$;

(4) *if* $k_2 > 2 + k_0$, *then* $y(x)$, $y'(x) \to +\infty$, $y''(x) \to 0$ *as* $x \to +\infty$.

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