On a Dirichlet Type Boundary Value Problem in an Orthogonally Convex Cylinder for a Class of Linear Partial Differential Equations

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Let $\Omega=(0,\omega_1)\times(0,\omega_2)\times(0,\omega_3)$ be an open rectangular box, and let

$$E=\big\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in\mathbf{D},\ x_3\in(0,\omega_3)\big\}$$

be an *orthogonally convex* cylinder with a piecewise smooth base inscribed in Ω. In view of the orthogonal convexity of the cylinder E, its base $\mathbf{D}$ admits the representations

$$\mathbf{D}=\big\{x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\big\}=\big\{x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\}.$$

In the domain E consider the boundary value problem

$$u^{(\mathbf{2})}=\sum_{\boldsymbol\alpha<\mathbf{2}}p_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)}+q(\mathbf{x}), \tag{1}$$

$$u(\eta_k(x_2),x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),$$
$$u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=\psi_3(x_1,x_2,(k-1)\omega_3)\ \ (k=1,2). \tag{2}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{2}=(2,2,2)$, $\boldsymbol\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index, $u^{(\boldsymbol\alpha)}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}$, $p_{\boldsymbol\alpha}\in C(\overline{E})$ $(\boldsymbol\alpha<\mathbf{2})$, $q\in C(\overline{E})$, $\psi_i\in C(\overline{E})$ $(i=1,2,3)$ and $\overline{E}$ is the closure of E.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u\in C^{2,2,2}(E)$ having continuous on $\overline{E}$ partial derivatives $u^{(2,0,0)}$ and $u^{(2,2,0)}$, and satisfying equation (1) and the boundary conditions (2) everywhere in E and ∂E, respectively.

Throughout the paper the following notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$, $\boldsymbol\alpha_i=(0,\dots,\alpha_i,\dots,0)$, $\boldsymbol\alpha_{ij}=\boldsymbol\alpha_i+\boldsymbol\alpha_j$.

$\boldsymbol\alpha=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol\beta=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol\alpha\neq\boldsymbol\beta$.

$\boldsymbol\alpha=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol\beta=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol\alpha<\boldsymbol\beta$, or $\boldsymbol\alpha=\boldsymbol\beta$.

$\|\boldsymbol\alpha\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

$\boldsymbol\Xi=\{\boldsymbol\sigma\big|\ \mathbf{0}<\boldsymbol\sigma<\mathbf{1}\}$.

$\boldsymbol\Upsilon_{\mathbf{2}}=\big\{\boldsymbol\alpha<\mathbf{2}:\ \alpha_i=2 \text{ for some } i\in\{1,2,3\}\big\}$.

$\boldsymbol{O}_{\mathbf{2}}=\big\{\boldsymbol\alpha<\mathbf{2}:\ \|\boldsymbol\alpha\| \text{ is odd}\big\}$.

$\operatorname{supp}\boldsymbol\alpha=\{i\mid\ \alpha_i>0\}$.

$\mathbf{x}_{\boldsymbol\alpha}=(\chi(\alpha_1)x_1,\chi(\alpha_2)x_2,\chi(\alpha_3)x_3)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.

$\widehat{\mathbf{x}}_{\boldsymbol\alpha}=\mathbf{x}-\mathbf{x}_{\boldsymbol\alpha}$.

$\mathbf{x}_{\boldsymbol\alpha}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\cdots i_l\}=\operatorname{supp}\boldsymbol\alpha$. Furthermore, $\mathbf{x}_{\boldsymbol\alpha}$ will be identified with $(\mathbf{x}_{\boldsymbol\alpha},\widehat{\mathbf{0}}_{\boldsymbol\alpha})$, and $\mathbf{x}$ will be identified with $(\mathbf{x}_{\boldsymbol\alpha},\widehat{\mathbf{x}}_{\boldsymbol\alpha})$, or with $(\mathbf{x}_{\boldsymbol\alpha},\mathbf{x}_{\widehat{\boldsymbol\alpha}})$.

$\Omega_{\boldsymbol\sigma}=[0,\omega_{i_1}]\times\cdots\times[0,\omega_{i_l}]$, where $\{i_1,\cdots i_l\}=\operatorname{supp}\boldsymbol\sigma$.

$\Omega_{ij}=(0,\omega_i)\times(0,\omega_j)$ $(1\le i<j\le3)$.

Along with problem (1), (2) consider the corresponding homogeneous problem

$$u^{(\mathbf{2})}=\sum_{\alpha<\mathbf{2}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

$$u(\eta_k(x_2),x_2,x_3)=0,\ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0,\ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0\ \ (k=1,2). \tag{2_0}$$

For each $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$ in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}\in\Omega_{\widehat{\boldsymbol{\sigma}}}$:

$$v^{(2,0,0)}=p_{022}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v+p_{122}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v^{(1,0,0)}, \tag{1_{100}}$$

$$v(\eta_1(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0,\quad v(\eta_2(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0; \tag{2_{100}}$$

$$v^{(0,2,0)}=p_{202}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v+p_{212}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v^{(0,1,0)}, \tag{1_{010}}$$

$$v(\gamma_1(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0,\quad v(\gamma_2(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0; \tag{2_{010}}$$

$$v^{(0,0,2)}=p_{220}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v+p_{221}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v^{(0,0,1)}, \tag{1_{001}}$$

$$v(0,\widehat{\mathbf{x}}_3)=0,\quad v(\omega_3,\widehat{\mathbf{x}}_3)=0; \tag{2_{001}}$$

$$v^{(\mathbf{2}_{12})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{12}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{12}}(\mathbf{x}_{12},\widehat{\mathbf{x}}_{12})v^{(\boldsymbol{\alpha})}, \tag{1_{110}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{12})=0,\quad v^{(2,0,0)}(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{12})=0\quad (k=1,2); \tag{2_{110}}$$

$$v^{(\mathbf{2}_{13})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{13}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{13}}(\mathbf{x}_{13},\widehat{\mathbf{x}}_{13})v^{(\boldsymbol{\alpha})}, \tag{1_{101}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{13})=0,\quad v^{(2,0,0)}((k-1)\omega_3,\widehat{\mathbf{x}}_{13})=0\ \ (k=1,2); \tag{2_{101}}$$

$$v^{(\mathbf{2}_{23})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{23}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{23}}(\mathbf{x}_{23},\widehat{\mathbf{x}}_{23})v^{(\boldsymbol{\alpha})}, \tag{1_{011}}$$

$$v(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{23})=0,\quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{23})=0\ \ (k=1,2). \tag{2_{011}}$$

Definition 1. Problem $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ $(\boldsymbol{\sigma}\in\boldsymbol{\Xi})$ is called $\boldsymbol{\sigma}$-associated problem of problem (1), (2).

Two-dimensional versions of problem (1), (2) were studied in [1], [2], where problems were considered in orthogonally convex smooth domains.

Orthogonal convexity of a domain is essential and cannot be relaxed. Examples attesting the paramount importance of the orthogonal convexity of a domain were introduced in Remarks 1 and 2 of [2]. Similar examples can be easily constructed for the three-dimensional case.

As follows from Remark 5 below, the C^2 regularity of functions η_k $(k=1,2)$ is essential for solvability of problem (1), (2) in a classical sense. However, C^2 regularity of functions η_k $(k=1,2)$ on the closed interval $[0,\omega_2]$ is impossible for smooth domains. Therefore we study the case of a piecewise smooth domain $\mathbf{D}$ separately from the case of a smooth domain $\mathbf{D}$. *Surprisingly*, some piecewise domains are better suited for the solvability of problem (1), (2), then domains with a C^∞ boundary.

Set

$$\mathbf{D}_{0,\delta}=[\eta_1(0)-\delta,\eta_1(0)+\delta]\times[0,\delta],\quad E_{0,\delta}=\mathbf{D}_{0,\delta}\times[0,\omega_3],$$
$$\mathbf{D}_{\omega_2,\delta}=[\eta_1(\omega_2)-\delta,\eta_1(\omega_2)+\delta]\times[\omega_2-\delta,\omega_2],\quad E_{\omega_2,\delta}=\mathbf{D}_{\omega_2,\delta}\times[0,\omega_3],$$
$$\varphi_{1k}(x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad \varphi_{2k}(x_1,x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),$$
$$\varphi_{3k}(x_1,x_2)=\psi_3(x_1,x_2,(k-1)\,\omega_3)\ \ (k=1,2). \tag{3}$$

Theorem 1. *Let*

$$\eta_k \in C^2([0,\omega_2]) \ \ (k=1,2), \tag{4}$$

$p_{\boldsymbol{\alpha}} \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C(\overline{\Omega})$, $\psi_1 \in C^{2,2,2}(\overline{E})$, $\psi_2 \in C^{0,2,2}(\overline{E})$, $\psi_1 \in C^{0,0,2}(\overline{E})$, *and let each* $\boldsymbol{\sigma}$*-associated problem* $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:*

- (i) *problem* $(1_0),(2_0)$ *has a finite dimensional space of solutions;*
- (ii) *if problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable, its solution belongs to* $C^{2,2,2}(\overline{E})$ *and admits the estimate*

$$\|u\|_{C^{2,2,2}(\overline{E})} \le M\bigg(\|q\|_{C(\overline{E})} + \sum_{k=1}^{2}\Big(\|\varphi_{1k}\|_{C^{2,2}(\overline{\Omega}_{2,3})} + \|\varphi_{2k}\|_{C^{0,2}(\overline{\Omega}_{1,3})} + \|\varphi_{3k}\|_{C(\overline{\mathbf{D}})}\Big)\bigg), \tag{5}$$

where M *is a positive constant independent of* φ_{1k}, φ_{2k}, φ_{3k} $(k=1,2)$ *and* q.

Theorem 2. *Let*

$$\gamma_k \in C^2((0,\omega_1)), \quad \eta_k \in C^2((0,\omega_2)) \ \ (k=1,2), \tag{6}$$

$$p_{\boldsymbol{\alpha}} \in C^{0,2,0}(\overline{E}) \ \ (\alpha_2 = 2, \ \boldsymbol{\alpha} < \mathbf{2}), \tag{7}$$

$p_{\boldsymbol{\alpha}} \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C(\overline{\Omega})$, $\psi_1 \in C^{2,2,2}(\overline{E})$, $\psi_2 \in C^{0,2,2}(\overline{E})$, $\psi_1 \in C^{0,0,2}(\overline{E})$, *and let each* $\boldsymbol{\sigma}$*-associated problem* $(1_{\boldsymbol{\sigma}}),(2_{\boldsymbol{\sigma}})$ *have only the trivial solution for every* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:* $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}} \in \Omega_{\widehat{\boldsymbol{\sigma}}}$ $(\boldsymbol{\sigma} \in \boldsymbol{\Xi})$. *Then problem* $(1),(2)$ *has the Fredholm property, i.e.:*

- (i) *problem* $(1_0),(2_0)$ *has a finite dimensional space of solutions;*
- (ii) *if problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable, its solution belongs to* $C^{2,2,2}(E)$ *and admits the estimate*

$$\begin{aligned}\|u\|_{C(\overline{E})} + \|u^{(2,0,0)}\|_{C(\overline{E})} + \|u^{(2,0,2)}\|_{C(\overline{E})}\qquad\qquad\\ \le M\bigg(\|q\|_{C(\overline{E})} + \sum_{k=1}^{2}\Big(\|\varphi_{1k}\|_{C^{0,2}(\overline{\Omega}_{2,3})} + \|\varphi_{2k}\|_{C^{0,2}(\overline{\Omega}_{1,3})} + \|\varphi_{3k}\|_{C(\overline{\mathbf{D}})}\Big)\bigg),\end{aligned} \tag{8}$$

where M *is a positive constant independent of* φ_{1k}, φ_{2k}, φ_{3k} $(k=1,2)$ *and* q.

Furthermore, if:

(F_1) $\mathbf{D}$ *is* ***strongly convex*** *near the points* $(\eta_1(0),0)$ *and* $\eta_2(\omega_2,\omega_2)$, *i.e.*

$$\gamma_1''(\eta_1(0)) > 0 \ \ \text{and} \ \ \gamma_2''(\eta_2(\omega_2)) < 0; \tag{9}$$

(F_2) $\gamma_1 \in C^5([\eta_1(0)-\delta,\eta_1(0)+\delta])$ *and* $\gamma_2 \in C^5([\eta_1(\omega_2)-\delta,\eta_1(\omega_2)+\delta])$ *for some* $\delta > 0$*;*

(F_3) $\psi_1 \in C^{5,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$*;*

(F_4) $\psi_2 \in C^{1,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$*;*

(F_5) $\psi_3 \in C^{3,0,0}(\mathbf{D}_{0,\delta} \cup \mathbf{D}_{\omega_2,\delta})$ *for some* $\delta > 0$*;*

(F_6) $p_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C^{3,0,0}(E_{0,\delta} \cup E_{\omega_2,\delta})$ *for some* $\delta > 0$, *then every solution of problem* (1), (2) *belongs to* $C^{2,2,2}(\overline{E})$.

Consider the equation

$$\begin{aligned} u^{(2,2,2)} = p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{022}(x_1)u^{(0,2,2)} \\ + p_{200}(x_2,x_3)u^{(2,0,0)} + p_{020}(x_1,x_3)u^{(0,2,0)} + p_{002}(x_1,x_2)u^{(0,0,2)} \\ + p_{201}(x_2)u^{(2,0,1)} + p_{102}(x_2)u^{(1,0,2)} + p_{021}(x_1)u^{(0,2,1)} + p_{012}(x_1)u^{(0,1,2)} \\ + p_{111}u^{(1,1,1)} + p_{100}(x_2,x_3)u^{(1,0,0)} + p_{010}(x_1,x_3)u^{(0,1,0)} + p_{001}(x_1,x_2)u^{(0,0,1)} \\ + p_{000}(x_1,x_2,x_3)u + q(x_1,x_2,x_3). \end{aligned} \tag{10}$$

Theorem 3. *Let condition* (4) *hold, let the domain* $\mathbf{D}$ *be convex, i.e.*

$$(-1)^{k-1}\eta_k''(x_2) \geq 0 \ \text{ for } \ x_2 \in (0,\omega_2) \ \ (k=1,2), \tag{11}$$

and let

$$p_{220}(x_3) \geq 0, \ \ p_{202}(x_2) \geq 0, \ \ p_{022}(x_1) \geq 0, \tag{12}$$

$$p_{200}(x_2,x_3) \leq 0, \ \ p_{020}(x_1,x_3) \leq 0, \ \ p_{002}(x_1,x_2) \leq 0, \tag{13}$$

$$p_{000}(x_1,x_2,x_3) \geq 0. \tag{14}$$

Then problem (10), (2) *is uniquely solvable, and its solution admits estimate* (5).

Theorem 4. *Let conditions* (6) *and inequalities* (11)–(14) *hold. Then problem* (10), (2) *is uniquely solvable, its solution belongs to* $C^{2,2,2}(\overline{\Omega})$ *and admits estimate* (8).
Furthermore, if conditions $(F_1)-(F_6)$ *hold, then the solution of problem* (1), (2) *belongs to* $C^{2,2,2}(\overline{E})$.

Remark 1. Condition (F_1) on the strong convexity of $\mathbf{D}$ is essential for the existence of a classical solution of problem (1), (2), and it cannot be replaced by strict convexity. Indeed, consider the problem

$$u^{(2,2,2)} = 0, \tag{15}$$

$$u(\eta_k(x_2),x_2,x_3)=0; \ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=2x_3^2; \ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0 \ \ (k=1,2) \tag{16}$$

in the domain $E = D \times (0,\omega_3)$, where $\mathbf{D} = \{(x_1,x_2) : \ (x_1-1)^4 + (x_2-1)^4 < 1\}$. It is clear that D is strictly convex, but not strongly convex, since

$$\gamma_k(x_1) = 1 + (-1)^k \sqrt[4]{1-(x_1-1)^4} \ \ (k=1,2), \quad \gamma_k''(x_1) > 0 \ \text{ for } \ x_1 \in (0,1) \cup (1,2),$$

and

$$\gamma_k''(1) = 0 \ \ (k=1,2).$$

As a result, the unique solution $u(\mathbf{x}) = \big((x_1-1)^2 - \sqrt{1-(x_2-1)^4}\,\big)x_3^2$ of problem (15), (16) does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,1,2)}$ is discontinuous along the rectangle $x_2 = 1$, $(x_1,x_3) \in [0,2] \times [0,\omega_3]$.

Remark 2. Consider the problem

$$u^{(2,2,2)} = 0, \tag{17}$$

$$\begin{aligned} u(\eta_k(x_2),x_2,x_3) = \psi_1(\eta_k(x_2),x_2,x_3); \quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3) = 0; \\ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3) = 0 \ \ (k=1,2) \end{aligned} \tag{18}$$

in the domain $E = D \times (0, \omega_3)$, where $\mathbf{D} = \{(x_1, x_2) : (x_1 - 1)^2 + (x_2 - 1)^2 < 1\}$, and

$$\psi_1(x_1, x_2, x_3) = \begin{cases} 0 & \text{for } 0 \le x_1 \le 1 \\ (x_1 - 1)^{4+\alpha} & \text{for } 1 \le x_1 \le 2 \end{cases}.$$

It is clear that D is strongly convex domain with the C^∞ boundary, and $\psi_1 \in C^4(\overline{E})$ but $\psi_1 \notin C^5(\overline{E})$ if $\alpha \in [0, 1)$. As a result, the unique solution of problem (17), (18)

$$u(\mathbf{x}) = \frac{x_1 - \eta_2(x_2)}{\eta_1(x_2) - \eta_2(x_2)}\, \psi_1\Big(1 + \sqrt{1 - (x_2 - 1)^2}\,\Big) = \frac{\sqrt{1 - (x_2 - 1)^2} - x_1}{2}\, \big(1 - (x_2 - 1)^2\big)^{2 + \frac{\alpha - 1}{2}}$$

does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segments $(1, 0, x_3)$ and $(1, 2, x_3)$, $x_3 \in [0, \omega_3]$.

Remark 3. Consider the problem

$$u^{(2,2,2)} = 0, \tag{19}$$

$$u(\eta_k(x_2), x_2, x_3) = 0; \;\; u^{(2,0,0)}(x_1, \gamma_k(x_1), x_3) = x_3^2; \;\; u^{(2,2,0)}(x_1, x_2, (k-1)\omega_3) = 0 \;\; (k = 1, 2) \tag{20}$$

in the domain $E = D \times (0, \omega_3)$, where $\mathbf{D}$ is a strongly convex C^2 domain inscribed in the rectangle $[0, 2] \times [0, 1]$ such that

$$\gamma_2(x_1) = 1 - (x_1 - 1)^2 + |x_1 - 1|^{4+\alpha} \;\text{ for }\; \frac{1}{2} < x_1 < \frac{3}{2}\,.$$

It is clear that if $\alpha \in [0, 1)$, then $\gamma_2 \in C^4([1 - \delta, 1 + \delta])$ but $\gamma_2 \notin C^5([1 - \delta, 1 + \delta])$ for any $\delta > 0$. Also,

$$\eta_1(x_2) = 1 + (1 - x_2)^{\frac{1}{2}} \Big(1 + c\,(1 - x_2)^{\frac{2+\alpha}{2}} + o((1 - x_2)^{\frac{3}{2}})\Big) \;\text{ for }\; x_2 \in [1 - \delta, 1],$$
$$\eta_1(x_2) = 1 - (1 - x_2)^{\frac{1}{2}} \Big(1 - c\,(1 - x_2)^{\frac{2+\alpha}{2}} + o((1 - x_2)^{\frac{3}{2}})\Big) \;\text{ for }\; x_2 \in [1 - \delta, 1]$$

for some $\delta > 0$, where c is a nonzero constant. As a result, problem (19), (20) has a unique solution

$$u(\mathbf{x}) = (x_1 - \eta_1(x_2))(x_2 - \eta_1(x_2)) = x^2 - x(\eta_1(x_2) + \eta_2(x_2)) - \eta_1(x_2)\eta_2(x_2),$$

which does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segment $(1, 1, x_3)$, $x_3 \in [0, \omega_3]$.

Remark 4. Consider the problem

$$u^{(2,2,2)} = 2|x_1 - 1|^\alpha \operatorname{sgn}(x_1 - 1), \tag{21}$$

$$u(\eta_k(x_2), x_2, x_3) = 0 \quad u^{(2,0,0)}(x_1, \gamma_k(x_1), x_3) = |x_1 - 1|^\alpha \operatorname{sgn}(x_1 - 1) x_3 (x_3 - \omega_3);$$
$$u^{(2,2,0)}(x_1, x_2, (k-1)\omega_3) = 0 \;\; (k = 1, 2) \tag{22}$$

in the domain $E = D \times (0, \omega_3)$, where $\mathbf{D}$ is a strongly convex C^2 domain inscribed in the rectangle $[0, 2] \times [0, 1]$ such that

$$\gamma_2(x_1) = 1 - (x_1 - 1)^2 \;\text{ for }\; \frac{1}{2} < x_1 < \frac{3}{2}\,.$$

It is clear that if $\alpha \in (2, 3)$, then $\psi_2(x_1, x_2, x_3) = |x_1|^\alpha \operatorname{sgn} x_1 x_3 (x_3 - \omega_3) \in C^{1,0,0}(E_{\omega_2,\delta} \cup E_{\omega_2,\delta})$ for some $\delta > 0$. Thus conditions $(F_1) - (F_5)$ hold, while condition (F_6) is violated.

As a result, problem (21), (22) has a unique solution

$$u(\mathbf{x}) = \frac{|x_1-1|^{2+\alpha}\operatorname{sgn}(x_1-1)-(x_1-1)(1-x_2)^{\frac{1+\alpha}{2}}}{(1+\alpha)(2+\alpha)}\, x_3(x_3-\omega_2) \ \text{ for } \ \frac{1}{2} < x_1 < \frac{3}{2}\,,$$

which does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,2,0)}$ and $u^{(1,2,0)}$ are discontinuous along the line segment $(1,1,x_3)$, $x_3 \in (0,\omega_3)$.

Remark 5. As we see, the functions γ_k $(k=1,2)$ can be piecewise smooth: γ_k may be nondifferentiable at points $\eta_1((k-1)\omega_2)$ and $\eta_2((k-1)\omega_2)$ (if they differ) $(k=1,2)$. On the other hand, C^2 smoothness of the functions η_k is essential and cannot be relaxed. Indeed, let $\alpha \in (1,2)$ be an arbitrary number,

$$\eta_k(x_2) = 1 + (-1)^k \sqrt{1 - \Big|x_2 - \frac{1}{2}\Big|^{\alpha}} \ \ (k=1,2),$$

and let u be a solution of the problem

$$u^{(2,2,2)} = 0, \tag{23}$$

$$u(\eta_k(x_2),x_2,x_3)=0; \ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=x_3^2; \ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0 \ \ (k=1,2). \tag{24}$$

Then

$$u^{(0,0,2)}(x_1,x_2,x_3) = x_1^2 - 2x_1 + |x_2-1|^{\alpha}.$$

Consequently, $u^{(0,1,2)}(x_1,x_2,x_3)$ is continuous on $\overline{E}$, however $u^{(0,2,2)}(x_1,x_2,x_3)$ is discontinuous along the line segment $0 \le x_1 \le 2$, $x_2 = 1$ since $\alpha \in (1,2)$. Thus, problem (23), (24) is not solvable in a classical sense due to the fact that the functions η_k are not twice differentiable at $x_2 = 1$.

References

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