Second Order Nonoscillatory Singular Linear Differential Equations

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In the present report, we give necessary and sufficient conditions for the nonoscillation and strong nonoscillation of second order singular linear homogeneous differential equations. These results are new for differential equations with continuous coefficients as well and generalize the classical results by Lyapunov [6], Hartman and Wintner [2], and Vallée Poussin [7]. They also generalize the theorems on the nonoscillation of singular differential equations given in the papers [1, 3, 5] and in our report [4].

On a finite open interval $]a,b[$, we consider the differential equation

$$u''(t)=p_1(t)u(t)+p_2(t)u'(t), \tag{1}$$

where $p_1,p_2:\,]a,b[\to\mathbb{R}$ are measurable functions, satisfying one of the following three conditions:

$$\int_a^b \big((t-a)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty, \tag{2_1}$$

$$\int_a^b \big((b-t)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty, \tag{2_2}$$

$$\int_a^b \big((t-a)(b-t)|p_1(t)|+|p_2(t)|\big)\,dt<+\infty. \tag{3}$$

We do not exclude the case, where

$$\int_a^b |p_1(t)|\,dt=+\infty,$$

i.e. the case when the function p_1 has nonintegrable singularity at least at one of the boundary points of the interval $]a,b[$. In such case equation (1) is said to be singular.

A continuously differentiable function $u:\,]a,b[\to\mathbb{R}$ is said to be **a solution to equation** (1) if its first derivative is absolutely continuous on every closed interval contained in $]a,b[$ and equation (1) is satisfied almost everywhere in $]a,b[$.

We assume that the values of a solution to equation (1) and its derivative at the points a and b are the corresponding one-sided limits of those functions if such limits exist.

It is well-known [5] that if condition (3) is satisfied, then any solution to equation (1) has finite right and left limits at the points a and b, and if the right limit (the left limit) of this solution is zero at the point a (at the point b), then its first derivative has a finite right (left) limit at the point a (at the point b).

Definition 1. Equation (1) is said to be **nonoscillatory** on the interval $[a,b]$ if its every solution has no more than one zero on that interval.

Definition 2. Equation (1) is said to be **strongly nonoscillatory from the right** (**strongly nonoscillatory from the left**) on the interval $[a,b]$ if for any $t_0 \in [a,b[$ (for any $t_0 \in]a,b]$), an arbitrary nontrivial solution u to equation (1), satisfying the condition

$$u(t_0)=0,$$

satisfies also the inequality

$$u'(t)\neq 0 \ \text{ for } \ t_0<t\leq b \quad \big(u'(t)\neq 0 \ \text{ for } \ a\leq t<t_0\big).$$

We use the following notation.

$$[x]_+=\frac{|x|+x}{2}\,, \quad [x]_-=\frac{|x|-x}{2} \ \text{ for } \ x\in\mathbb{R}.$$

If $w:]a,b[\to \mathbb{R}$ is a differentiable function, then

$$\begin{aligned} h_1(p_1,p_2,w)(t) &= [p_1(t)]_-w(t)+[p_2(t)]_-w'(t),\\ h_2(p_1,p_2,w)(t) &= [p_1(t)]_-w(t)-[p_2(t)]_+w'(t). \end{aligned}$$

Theorem 1_1. *Let condition* (2_1) *hold and let there exist a continuously differentiable function* $w:[a,b]\to[0,+\infty[$ *such that*

$$w(a)=0, \ \ w'(t)>0, \ \int\limits_t^b h_1(p_1,p_2,w)(s)\,ds\leq w'(t) \ \textit{ for } \ a\leq t<b, \tag{4_1}$$

$$\limsup_{t\to b}\int\limits_t^b \frac{h_1(p_1,p_2,w)(s)}{w'(t)}\,ds<1. \tag{5_1}$$

Then the differential equation (1) *is strongly nonoscillatory from the right on* $[a,b]$.

Theorem 1_2. *Let condition* (2_2) *hold and let there exist a continuously differentiable function* $w:[a,b]\to[0,+\infty[$ *such that*

$$w(b)=0, \ \ w'(t)<0, \ \int\limits_a^t h_2(p_1,p_2,w)(s)\,ds\leq |w'(t)| \ \textit{ for } \ a<t\leq b, \tag{4_2}$$

$$\limsup_{t\to a}\int\limits_a^t \frac{h_2(p_1,p_2,w)(s)}{|w'(t)|}\,ds<1. \tag{5_2}$$

Then the differential equation (1) *is strongly nonoscillatory from the left on* $[a,b]$.

Theorem 2_1. *Let along with condition* (2_1) *the condition*

$$p_1(t) \leq 0, \quad p_2(t) \leq 0 \ \ \textit{for} \ \ a < t < b \tag{6_1}$$

hold. Then for the differential equation (1) *to be strongly nonoscillatory from the right on* $[a, b]$*, necessary and sufficient is the existence of such a continuously differentiable function* $w : [a, b] \to [0, +\infty[$*, which satisfies conditions* (4_1) *and* (5_1)*.*

Theorem 2_2. *Let along with condition* (2_2) *the condition*

$$p_1(t) \leq 0, \quad p_2(t) \geq 0 \ \ \textit{for} \ \ a < t < b \tag{6_2}$$

hold. Then for the differential equation (1) *to be strongly nonoscillatory from the left on* $[a, b]$*, necessary and sufficient is the existence of such a continuously differentiable function* $w : [a, b] \to [0, +\infty[$*, which satisfies conditions* (4_2) *and* (5_2)*.*

Theorems 1_1 and 1_2 yield unimprovable effective conditions guaranteeing the strong nonoscillation from the right and left of the differential equation (1) on the interval $[a, b]$. Namely, the following statements are valid.

Corollary 1_1. *Let along with condition* (2_1) *one of the following three conditions hold:*

$$\int\limits_a^b \big((t-a)[p_1(t)]_- + [p_2(t)]_-\big)\, dt \leq 1, \tag{7_1}$$

$$p_1(t) \geq -\frac{\lambda_1 (t-a)^\alpha}{(\alpha+3)(b-a)^{\alpha+2} - (t-a)^{\alpha+2}}\,, \quad p_2(t) \geq -\lambda_2 (t-a)^{\alpha+1} \ \ \textit{for} \ \ a < t < b, \tag{8_1}$$

$$p_1(t) \geq -\ell_1, \quad p_2(t) \geq -\ell_2 \ \ \textit{for} \ \ a < t < b, \tag{9_1}$$

where $\alpha > -2$*, while* λ_i *and* ℓ_i $(i = 1, 2)$ *are nonnegative constants such that*

$$\frac{\lambda_1}{\alpha+3} + (b-a)^{\alpha+2} \lambda_2 < \alpha + 2, \tag{10}$$

$$\int\limits_0^{+\infty} \frac{dx}{\ell_1 + \ell_2 x + x^2} > b - a. \tag{11}$$

Then the differential equation (1) *is strongly nonoscillatory from the right on* $[a, b]$*.*

Corollary 1_2. *Let along with condition* (2_2) *one of the following three conditions hold:*

$$\int\limits_a^b \big((b-t)[p_1(t)]_- + [p_2(t)]_+\big)\, dt \leq 1, \tag{7_2}$$

$$p_1(t) \geq -\frac{\lambda_1 (b-t)^\alpha}{(\alpha+3)(b-a)^{\alpha+2} - (b-t)^{\alpha+2}}\,, \quad p_2(t) \leq \lambda_2 (b-t)^{\alpha+2} \ \ \textit{for} \ \ a < t < b, \tag{8_2}$$

$$p_1(t) \geq -\ell_1, \quad p_2(t) \leq \ell_2 \ \ \textit{for} \ \ a < t < b, \tag{9_2}$$

where $\alpha > -2$*, while* λ_i *and* ℓ_i $(i = 1, 2)$ *are nonnegative constants satisfying inequalities* (10) *and* (11)*. Then the differential equation* (1) *is strongly nonoscillatory from the left on* $[a, b]$*.*

Remark 1. In the right-hand sides of inequalities (7_1) and (7_2), 1 cannot be replaced by $1 + \varepsilon$ no matter how small $\varepsilon > 0$ is, and the strict inequalities (10) and (11) cannot be replaced by the non-strict ones.

Theorem 3. *Let condition* (3) *hold and there exist a number* $t_0 \in \,]a,b[$ *and continuously differentiable functions* $w_1 : [a,t_0] \to [0,+\infty[\,$, $w_2 : [t_0,b] \to [0,+\infty[$ *such that*

$$w_1(a) = 0, \quad w_1'(t) > 0, \quad \int_t^{t_0} h_1(p_1,p_2,w_1)(s)\,ds \le w_1'(t) \;\; \text{for } a \le t < t_0, \tag{12}$$

$$w_2(b) = 0, \quad w_2'(t) < 0, \quad \int_{t_0}^{t} h_2(p_1,p_2,w_2)(s)\,ds < |w_2'(t)| \;\; \text{for } t_0 < t \le b, \tag{13}$$

$$\limsup_{t\to t_0} \int_t^{t_0} \frac{h_1(p_1,p_2,w_1)(s)}{w_1'(t)}\,ds + \limsup_{t\to t_0} \int_{t_0}^{t} \frac{h_2(p_1,p_2,w_2)(s)}{|w_2'(t)|}\,ds < 2. \tag{14}$$

Then the differential equation (1) *is nonoscillatory on* $[a,b]$.

Theorem 4. *If*

$$p_1(t) \le 0, \quad p_2(t) = 0 \;\; \text{for } a < t < b, \quad \int_a^b (t-a)|p_1(t)|\,dt < +\infty,$$

then for the differential equation (1) *to be nonoscillatory on* $[a,b]$, *necessary and sufficient is the existence of such a number* $t_0 \in \,]a,b[$ *and continuously differentiable functions* $w_1 : [a,t_0] \to [0,+\infty[\,$, $w_2 : [t_0,b] \to [0,+\infty[\,$, *which satisfy conditions* (12)–(14).

Corollary 2. *Let along with inequality* (3), *for some* $t_0 \in \,]a,b[$ *one of the following three conditions hold:*

$$\int_a^{t_0} \big((t-a)[p_1(t)]_- + [p_2(t)]_-\big)\,dt \le 1, \quad \int_{t_0}^{b} \big((b-t)[p_1(t)]_- + [p_2(t)]_+\big)\,dt \le 1, \tag{15}$$

$$\begin{gathered} p_1(t) \ge -\frac{\lambda_{11}}{(2t_0-a-t)(t-a)}\,, \quad p_2(t) \ge -\lambda_{12} \;\; \text{for } a < t < t_0, \\ p_1(t) \ge -\frac{\lambda_{21}}{(b+t-2t_0)(b-t)}\,, \quad p_2(t) \le \lambda_{22} \;\; \text{for } t_0 < t < b, \end{gathered} \tag{16}$$

$$p_1(t) \ge -\ell_{11}, \;\; p_2(t) \ge -\ell_{12} \;\; \text{for } a < t < t_0, \quad p_1(t) \ge -\ell_{21}, \;\; p_2(t) \le \ell_{22} \;\; \text{for } t_0 < t < b, \tag{17}$$

where λ_{ik} *and* ℓ_{ik} $(i,k = 1,2)$ *are nonnegative constants such that*

$$\lambda_{11} + 2(t_0-a)\lambda_{12} < 2, \quad \lambda_{21} + 2(b-t_0)\lambda_{22} < 2, \tag{18}$$

$$\int_0^{+\infty} \frac{dx}{\ell_{11} + \ell_{12}x + x^2} > t_0 - a, \quad \int_0^{+\infty} \frac{dx}{\ell_{21} + \ell_{22}x + x^2} > b - t_0. \tag{19}$$

Then the differential equation (1) *is nonoscillatory on* $[a,b]$.

Remark 2. In the right-hand sides of inequalities (15), 1 cannot be replaced by $1+\varepsilon$ no matter how small $\varepsilon > 0$ is, and the strict inequalities (18) and (19) cannot be replaced by the non-strict ones.

If

$$\int_a^b \left(\frac{(t-a)(b-t)}{b-a} \, [p_1(t)]_- + |p_2(t)| \right) dt \le 1, \tag{20}$$

then there exists $t_0 \in \,]a,b[$ such that inequalities (15) are satisfied.

On the other hand, it is obvious that if for some nonnegative constants ℓ_1 and ℓ_2 the inequalities

$$p_1(t) \ge -\ell_1, \quad |p_2(t)| \le \ell_2 \ \text{ for } \ a < t < b, \tag{21}$$

$$\int_0^{+\infty} \frac{dx}{\ell_1 + \ell_2 x + x^2} > (b-a)/2 \tag{22}$$

are satisfied, then inequalities (17) and (19) hold as well, where $\ell_{11} = \ell_{21} = \ell_1$, $\ell_{12} = \ell_{22} = \ell_2$, $t_0 = (a+b)/2$. Therefore, the following statements are valid.

Corollary 3. *If*

$$\int_a^b (t-a)(b-t)|p_1(t)| \, dt < +\infty \tag{23}$$

and inequality (20) *holds, then the differential equation* (1) *is nonoscillatory on* $[a,b]$.

Corollary 4. *Let there exist nonnegative constants* ℓ_1 *and* ℓ_2 *such that along with* (23) *conditions* (21) *and* (22) *are satisfied. Then the differential equation* (1) *is nonoscillatory on* $[a,b]$.

In the case, where p_1 and p_2 are continuous on $[a,b]$ functions, Corollary 3 implies the theorems by Lyapunov [6] and Hartman–Wintner [2], while Corollary 4 yields the Vallée Poussin theorem [7].

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