Optimal Conditions for the Unique Solvability of Two-Point Boundary Value Problems for Third Order Linear Singular Differential Equations

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On a finite open interval $]a,b[\,$, we consider the linear differential equation

$$u''' = p(t)u + q(t) \tag{1}$$

with the boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad \sum_{i=0}^{k} \ell_i u^{(i)}(b-) = 0. \tag{2}$$

Here

$$k \in \{0,1,2\}, \;\; \ell_i \geq 0 \;\; (i = 0,\dots,k), \;\; \ell_k > 0,$$

while p and $q :]a,b[\to \mathbb{R}$ are measurable functions such that

$$\int_a^b (t-a)^2(b-t)^{2-k}|p(t)|\,dt < +\infty, \quad \int_a^b (t-a)(b-t)^{2-k}|q(t)|\,dt < +\infty. \tag{3}$$

We are mainly interested in the case where the functions p and q have nonintegrable singularities at the boundary points of the interval $]a,b[\,$, i.e. the case, where

$$\int_a^b \big(|p(t)| + |q(t)|\big)\,dt = +\infty.$$

However, the results below on the unique solvability of problem $(1),(2)$ are new also for the regular case when the functions p and q are integrable on $[a,b]$.

To formulate the above mentioned results, we need the following notation.

$$\Delta_k(t) = \sum_{i=0}^{k} \frac{(b-t)^{2-i}}{(2-i)!}\,\ell_i \Big/ \sum_{i=0}^{k} \frac{(b-a)^{2-i}}{(2-i)!}\,\ell_i,$$

$$g_k(t,s) = \begin{cases} \dfrac{1}{2}\big(\Delta_k(s)(t-a)^2 - (t-s)^2\big) & \text{for } a \leq s < t \leq b, \\[2mm] \dfrac{1}{2}\,\Delta_k(s)(t-a)^2 & \text{for } a \leq t \leq s \leq b, \end{cases}$$

$$r_0(\alpha) = 1, \quad r_1(\alpha) = \frac{\ell_0(b-a) + (\alpha+3)\ell_1}{\ell_0(b-a) + 2\ell_1}\,,$$

$$r_2(\alpha)=\frac{\ell_0(b-a)^2+(\alpha+3)\ell_1(b-a)+(\alpha+3)(\alpha+2)\ell_2}{\ell_0(b-a)^2+2\ell_1(b-a)+2\ell_2}\,,$$

$$p_k(t;\alpha)=\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{r_k(\alpha)(b-a)^{\alpha+1}-(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2}\ \text{ for }\ 0<t<b,\ \ \alpha>-1,$$

$$p_-(t)\equiv\big(|p(t)|-p(t)\big)/2.$$

In [1] it is stated that problem $(1),(2)$ is uniquely solvable if and only if the homogeneous problem

$$u'''=p(t)u \tag{1_0}$$

under the boundary conditions (2) has only a trivial solution. Based on this fact the following theorem is proved.

Theorem. *Let there exist a continuous function $w:\,]a,b[\,\to\,]a,b[$ such that along with (3) the following conditions*

$$\sup\bigg\{\int\limits_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\bigg\}<1, \tag{4}$$

$$\liminf_{t\to a}\frac{w(t)}{(t-a)^2}>0,\quad \liminf_{t\to b}\frac{w(t)}{(b-t)^{m_k}}>0 \tag{5}$$

hold, where $m_k=(1-k+|1-k|)/2$. Then problem $(1),(2)$ has a unique solution.

Corollary 1. *If for some $\alpha>-1$ along with (3) the conditions*

$$p(t)\geq -p_k(t;\alpha)\ \ \textit{for}\ \ a<t<b, \tag{6}$$

$$\operatorname{mes}\big\{t\in\,]a,b[\,:\ p(t)>-p_k(t;\alpha)\big\}>0 \tag{7}$$

hold, then problem $(1),(2)$ has a unique solution.

Corollary 2. *If along with (3) the condition*

$$\int\limits_a^b(t-a)^2\Delta_k(t)p_-(t)\,dt<2 \tag{8}$$

holds, then problem $(1),(2)$ has a unique solution.

Remark 1. In the above formulated theorem, inequality (4) is unimprovable and it cannot be replaced by the nonstrict inequality

$$\sup\bigg\{\int\limits_a^b\frac{g_k(t,s)}{w(t)}\,w(s)p_-(s)\,ds:\ a<t<b\bigg\}\leq 1. \tag{9}$$

Indeed, if

$$p(t)\equiv -p_k(t;\alpha),\quad w(t)\equiv\big(r_k(\alpha)(b-a)^{\alpha+1}-(b-t)^{\alpha+1}\big)(t-a)^2,$$

where $\alpha>-1$, then inequalities (5) are satisfied, while inequality (4) is violated instead of which inequality (9) holds. On the other hand, in this case the homogeneous problem $(1_0),(2)$ has a nontrivial solution $u(t)\equiv w(t)$ and, consequently, problem $(1),(2)$ is not uniquely solvable no matter how the function q is.

Remark 2. The strict inequality (7) in Corollary 1 cannot be replaced by the nonstrict one since if $p(t) \equiv -p_k(t;\alpha)$, then the homogeneous problem $(1_0), (2)$ has a nontrivial solution.

Remark 3. In the case, where $k \in \{1,2\}$, the strict inequality (8) in Corollary 2 cannot be replaced by the condition

$$\int_a^b (t-a)^2 \Delta_k(t) p_-(t)\,dt < 2+\varepsilon \tag{10}$$

no matter how small $\varepsilon > 0$ is. Indeed, if $p(t) \equiv -p_k(t;\alpha)$ and $\alpha > 0$ is so large that

$$r_k(\alpha) > 1 + \frac{2}{\varepsilon}\,,$$

then inequality (8) is violated but inequality (9) holds. On the other hand, as we already mentioned above, in this case the homogeneous problem $(1_0), (2)$ has a nontrivial solution.

Particular cases of the boundary conditions (2) are the Dirichlet boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad u(b-) = 0, \tag{2_0}$$

and the Nicoletti boundary conditions

$$u(a+) = 0, \quad u'(a+) = 0, \quad u'(b-) = 0, \tag{2_1}$$

$$u(a+) = 0, \quad u'(a+) = 0, \quad u''(b-) = 0. \tag{2_2}$$

For problem $(1), (2_k)$ $(k = 0,1,2)$, a pair of conditions (6), (7) has one of the following three forms:

$$p(t) \geq -\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(b-a)^{\alpha+1} - (t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_0}$$

$$\operatorname{mes}\Big\{t \in \,]a,b[\,:\ (t-a)^{2-\alpha} p(t) > -\frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(b-a)^{\alpha+1} - (t-a)^{\alpha+1}}\Big\} > 0; \tag{7_0}$$

$$p(t) \geq -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+3)(b-a)^{\alpha+1} - 2(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_1}$$

$$\operatorname{mes}\Big\{t \in \,]a,b[\,:\ (t-a)^{2-\alpha} p(t) > -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+3)(b-a)^{\alpha+1} - 2(t-a)^{\alpha+1}}\Big\} > 0; \tag{7_1}$$

$$p(t) \geq -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+2)(\alpha+3)(b-a)^{\alpha+1} - 2(t-a)^{\alpha+1}}\,(t-a)^{\alpha-2} \ \text{ for } \ a<t<b, \tag{6_2}$$

$$\operatorname{mes}\Big\{t \in \,]a,b[\,:\ (t-a)^{2-\alpha} p(t) > -\frac{2(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+2)(\alpha+3)(b-a)^{\alpha+1} - 2(t-a)^{\alpha+1}}\Big\} > 0. \tag{7_2}$$

Corollary 3. *Let for some $k \in \{0,1,2\}$ along with* (3) *conditions* (6_k) *and* (7_k) *be satisfied. Then problem* $(1), (2_k)$ *has a unique solution.*

Corollary 4. *If for some $k \in \{0,1,2\}$ along with* (3) *the condition*

$$\int_a^b (t-a)^2 (b-t)^{2-k} p_-(t)\,dt < 2(b-a)^{2-k} \tag{11}$$

is satisfied, then problem $(1), (2_k)$ *has a unique solution.*

Remark 4. The strict inequality (7_k) in Corollary 3 cannot be replaced by the nonstrict one, while inequality (11) in Corollary 4 for some $k \in \{1, 2\}$ cannot be replaced by the inequality

$$\int\limits_a^b (t-a)^2 (b-t)^{2-k} p_-(t)\, dt < (2+\varepsilon)(b-a)^{2-k}$$

no matter how small $\varepsilon > 0$ is.

References

[1] I. Kiguradze, On the unique solvability of two-point boundary value problems for third order linear differential equations with singularities. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 3, 375–390.