Approximate Solution of Optimal Control Problem for Differential Inclusion with Fast-Oscillating Coefficients on Semi-Axis

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1 Introduction

There are many approaches for investigation of control problems for differential equations and inclusions, in particular, asymptotic methods are widely applied. It is worth to emphasize the averaging method, for which M. M. Krylov and M. M. Bogolyubov proposed the strict mathematical justification. In works of V. A. Plotnikov and works of his school (see, for example, [12]) there is the strict justification of the averaging method in application to control problems. In monograph [13] one ca find a justification of the averaging method, in particular, for ordinary differential inclusions, partial differential inclusions, inclusions with Hukuhara derivative. In the paper [11] the time averaging was performed firstly, where time is clearly included in the system, at that the control function was considered a parameter and averaging was not performed on it. Moreover, the authors had to impose a condition of asymptotic stability for the control function. In the paper [5] the approach from [11] is applied to the solvability of the optimal control problem on finite interval, but however, the rather strict condition of asymptotic stability is removed. In the paper [6] similar results to [5] are obtained on semi-axis. In the paper [7] authors apply the averaging method to solve the optimal control problem with fast-oscillating variables which is linear by control on a finite interval; at that the system of differential inclusions with Lipschitz right-hand side by phase variable. Optimal control problems on semi-axis in different perturbed problems are studied in [3, 4, 8–11, 14, 15].

In this work we apply the averaging method to investigate the optimal control problem with fast oscillating variables for the system of differential inclusions on semi-axis. In particular, we prove the solvability of original problem as well as averaged problem using the direct method of calculus of variations. We justify the convergence of optimal controls and optimal trajectories of solutions of original problem to optimal control and optimal trajectory of solutions of averaged problem. We show that optimal control of averaged problem is asymptotically optimal for the original exact problem.

2 Statement of the problem and the main results

Let us consider an optimal control problem for the system of differential inclusions on semi-axis with a small parameter and fast-oscillating coefficients

$$\dot{x} \in f\Big(\frac{t}{\varepsilon}, x\Big) + f_1(x)u(t), \quad x(0, u(0)) = x_0 \tag{2.1}$$

with the quality criterion

$$J_\varepsilon[x,u]=\int\limits_0^\infty \big(e^{-jt}A(t,x_\varepsilon(t))+u^2(t)\big)\,dt\to\inf. \tag{2.2}$$

Here $\varepsilon>0$ is a small parameter, $j>0$ is a fixed constant that defines discount, x is a phase vector in $\mathbb{R}^d$, $u(t)$ is m-measurable control vector which takes values in some set $U\subset\mathbb{R}^m$.

Let there be an uniformly by $x\in\mathbb{R}^d$ averaged value for a multi-valued function

$$\lim_{s\to\infty}\frac{1}{s}\int\limits_0^s f(t,x)\,dt=f_0(x), \tag{2.3}$$

where the integral of multi-valued function we consider in the sense of Aumann [1], and the limit of multi-valued function we consider in the sense of Hausdorf.

The optimal control problem on semi-axis (2.1), (2.2) is matched by the average control problem:

$$\dot y\in f_0(y)+f_1(y)u(t),\quad y(0,u(0))=x_0 \tag{2.4}$$

with the quality criterion

$$J_0[x,u]=\int\limits_0^\infty \big(e^{-jt}A(t,y(t))+u^2(t)\big)\,dt\to\inf. \tag{2.5}$$

Let for the problem (2.1), (2.2) and the corresponding average problem (2.4), (2.5) the next conditions are satisfied:

Condition 2.1. We consider m-measurable vector-functions $u(\cdot)\in L_2([0,\infty))$, which takes values in closed convex set $U\subset\mathbb{R}^m$ as admissible controls, and we consider that $0\in U$ as well.

Condition 2.2. The function $A(t,s)$ is defined for $t\ge 0, x\in\mathbb{R}^d, u\in U$, measurable by t and continuous by x, at that

$$\exists\, C>0:\ \ A(t,x)\ge -C,$$

and satisfies the next growth condition by $x\in\mathbb{R}^d$:

$$\exists\, K>0:\ \ |A(t,x)|\le K\big(1+|x|^p\big)$$

for each $t\ge 0$ and $x\in\mathbb{R}^d$, $p\ge 0$.

Condition 2.3. The multi-valued function $f(t,x)$ ($f:Q=\{t\ge 0,\ x\in\mathbb{R}^d\}\to conv(\mathbb{R}^d)$) is defined and continuous in the Hausdorf metrics over the set of variables in Q, and matrix-valued function $f_1(x)$ is continuous by $x\in\mathbb{R}^d$ and the next conditions are fulfilled:

(1) $f(t,x)$ satisfies the linear growth condition by x with constants L_1 and L_2 in the domain Q, namely,

$$\|f(t,x)\|_+:=\sup_{\xi\in F(t,x)}\|\xi\|\le L_1+L_2|x|\qquad \forall (t,x)\in Q;$$

$f_1(x)$ satisfies the linear growth condition by x in the domain $\mathbb{R}^d$ with constants L_3 and L_4, namely,

$$|f_1(x)|\le L_3+L_4|x|,$$

where

$$j>L_2p; \tag{2.6}$$

(2) $f(t,x)$ and $f_1(x)$ satisfy Lipschitz condition by x uniformly by t in the definition domain with $K_1, K_2 > 0$, respectively.

Condition 2.4. The averaged value of multi-valued function f in the sense of limit (2.3) is a single-valued continuous function.

Taking into account conditions for parameters of problem we can obtain the result concerning solvability of problem (2.1), (2.2). Namely, we have the next

Lemma 1. *Under Conditions 2.1, 2.2, 2.3 there exists a solution of the optimal control problem* (2.1), (2.2).

Taking into account the previous Lemma, we can obtain the similar result about the solvability of the averaged problem (2.4), (2.5).

In the next result we show the convergence of optimal controls, optimal trajectories and optimal values of quality criterion of the original problem (2.1), (2.2) to corresponding parameters of the averaged problem (2.4), (2.5).

Theorem 1. *Let* $(x_\varepsilon^*(t), u_\varepsilon^*(t))$ *be the solution of the problem* (2.1), (2.2). *Then for some solution* $(y^*(t), u^*(t))$ *of problem* (2.4), (2.5) *we have:*

(1) $J_\varepsilon^* \to J_0^*$, $\varepsilon \to 0$ *and* $J_\varepsilon^* = \inf\limits_{x,u} \in \Xi_1 J_\varepsilon[x,u]$, $J_0^* = \inf\limits_{(x,u)\in\Xi_2} J_0[x,u]$, Ξ_1, Ξ_2 *are sets of admissible pairs for problems* (2.1), (2.2) *and* (2.4), (2.5), *respectively.*

(2) *for each* $\eta > 0$ *there exists* $\varepsilon_0 = \varepsilon_0(\eta)$ *such that* $0 < \varepsilon < \varepsilon_0$ *we have*

$$\big|J_\varepsilon^* - J[x_\varepsilon^*, u^*]\big| < \eta, \tag{2.7}$$

where x_ε^* *is the solution of Cauchy problem* (2.1);

(3) *there exists a sequence* $\varepsilon_n \to 0$, $n \to \infty$ *such that*

$$x_{\varepsilon_n}^* \to y(t) \tag{2.8}$$

uniformly on each interval $[0,T]$ *for any* $T > 0$, *and*

$$u_{\varepsilon_n}^* \xrightarrow{w} u^* \tag{2.9}$$

weakly in $L_2([0,\infty))$.

If, moreover, there exists a unique solution of the averaged problem (2.4), (2.5), *then the convergences* (2.8), (2.9) *take place for all* $\varepsilon \to 0$.

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References

[1] R. J. Aumann, Integrals of set-valued functions. *J. Math. Anal. Appl.* **12** (1965), 1–12.

[2] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides.* (Russian) "Nauka", Moscow, 1985.

[3] V. O. Kapustyan, O. A. Kapustyan and O. K. Mazur, An optimal control problem for the Poisson equation with nonlocal boundary conditions. (Ukrainian) *Nelīnīĭnī Kolıv.* **16** (2013), no. 3, 350–358; translation in *J. Math. Sci. (N.Y.)* **201** (2014), no. 3, 325–334.

[4] O. V. Kapustyan, O. A. Kapustyan and A. V. Sukretna, Approximate stabilization for a nonlinear parabolic boundary-value problem. *Ukrainian Math. J.* **63** (2011), no. 5, 759–767.

[5] O. D. Kīchmarenko, Application of the averaging method to optimal control problems for ordinary differential equations on the semiaxis. *Int. J. Pure Appl. Math.* **115** (2017), no. 1, 93–114.

[6] O. D. Kīchmarenko, Application of the averaging method to optimal control problems for ordinary differential equations on the semiaxis. (Ukrainian) *Ukraïn. Mat. Zh.* **70** (2018), no. 5, 642–654; translation in *Ukrainian Math. J.* **70** (2018), no. 5, 739–753.

[7] O. D. Kichmarenko, N. V. Kasimova and T. Yu Zhuk, Approximate solution of optimal control problem for differential inclusion with fast-oscillating coefficients. (Ukrainian) *Res. in Math. and Mech.* **26** (2021) (*in print*).

[8] O. Kichmarenko and O. Stanzhytskyi, Sufficient conditions for the existence of optimal controls for some classes of functional-differential equations. *Nonlinear Dyn. Syst. Theory* **18** (2018), no. 2, 196–211.

[9] O. Lavrova, V. Mogylova, O. Stanzhytskyi and O. Misiats, Approximation of the optimal control problem on an interval with a family of optimization problems on time scales. *Nonlinear Dyn. Syst. Theory* **17** (2017), no. 3, 303–314.

[10] O. G. Nakonechniĭ, O. A. Kapustyan and A. O. Chikrīĭ, Approximate guaranteed mean-square estimates for functionals from solutions of parabolic problems with rapidly oscillating coefficients under nonlinear observations. (Ukrainian) *Kibernet. Sistem. Anal.* **55** (2019), no. 5, 95–105; translation in *Cybernet. Systems Anal.* **55** (2019), no. 5, 785–795.

[11] T. V. Nosenko and O. M. Stanzhits'kiĭ, The averaging method in some optimal control problems. (Ukrainian) *Nelīnīĭnī Kolıv.* **11** (2008), no. 4, 512–519 (2009); translated in *Nonlinear Oscil. (N. Y.)* **11** (2008), no. 4, 539–547.

[12] V. A. Plotnikov, *The Averaging Method in Control Problems.* (Russian) 'Lybid', Kiev, 1992.

[13] V. A. Plotnikov, A. V. Plotnikov and A. N. Vityuk, *Differential Equations with a Multivalued Right-Hand Side. Asymptotic Methods.* (Russian) "AstroPrint", Odessa, 1999.

[14] O. M. Stanzhyts'kyi, Investigation of exponential dichotomy of itô stochastic systems by using quadratic forms. *Ukrainian Math. J.* **53** 2001, 1882 -1894.

[15] N. V. Zadoyanchuk and P. O. Kas'yanov, The Faedo–Galerkin method for second-order evolution inclusions with W_λ-pseudomonotone mappings. (Ukrainian) *Ukraïn. Mat. Zh.* **61** (2009), no. 2, 195–213; translated in *Ukrainian Math. J.* **61** (2009), no. 2, 236–258.