The Boundary Value Problem for a Semilinear Hyperbolic System

Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mail: kharibegashvili@yahoo.com

In the space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,\dots,x_n)$ and t, we consider the semilinear hyperbolic system of the form

$$\square^2 u_i + f_i(u_1,u_2,\dots,u_N) = F_i(x,t),\ i=1,\dots,N, \tag{1}$$

where $f=(f_1,\dots,f_N)$, $F=(F_1,\dots,F_N)$ are the given, and $u=(u_1,\dots,u_N)$ is an unknown real vector function, $n\geq 2$, $N\geq 2$, $\square := \frac{\partial^2}{\partial t^2}-\Delta$, $\Delta := \sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$.

For the system (1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, a solution $u=u(x,t)$ of that system according to the boundary conditions

$$u\big|_{\partial D_T}=0, \quad \frac{\partial u}{\partial\nu}\Big|_{\partial D_T}=0, \tag{2}$$

where $\nu=(\nu_1,\dots,\nu_n,\nu_{n+1})$ is the unit vector of the outer normal to ∂D_T.

Let

$$\overset{\circ}{C}{}^k(\overline{D}_T,\partial D_T) := \Big\{u\in C^k(\overline{D}_T):\ u\big|_{\partial D_T}=\frac{\partial u}{\partial\nu}\Big|_{\partial D_T}=0\Big\},\ \ k\geq 2.$$

Assume $u\in \overset{\circ}{C}{}^4(\overline{D}_T,\partial D_T)$ is a classical solution of the problem (1), (2). Multiplying both parts of the system (1) scalarly by an arbitrary vector function $\varphi=(\varphi_1,\dots,\varphi_N)\in \overset{\circ}{C}{}^2(\overline{D}_T,\partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$\int\limits_{D_T}\square u\,\square\varphi\,dx\,dt + \int\limits_{D_T} f(u)\varphi\,dx\,dt = \int\limits_{D_T} F\varphi\,dx\,dt. \tag{3}$$

When deducing (3), we have used the equality

$$\int\limits_{D_T}\square u\,\square\varphi\,dx\,dt = \int\limits_{\partial D_T}\frac{\partial\varphi}{\partial N}\square u\,ds - \int\limits_{\partial D_T}\varphi\,\frac{\partial}{\partial N}\square u\,ds + \int\limits_{D_T}\varphi\square^2 u\,dx\,dt,$$

where

$$\frac{\partial}{\partial N} = \nu_{n+1}\frac{\partial}{\partial t} - \sum_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}$$

is the derivative with respect to the conormal, as well as the equalities

$$\frac{\partial\varphi}{\partial N}\Big|_{\Gamma} = -\frac{\partial\varphi}{\partial\nu}\Big|_{\Gamma}, \quad \frac{\partial\varphi}{\partial N}\Big|_{\partial D_T\setminus\Gamma} = \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D_T\setminus\Gamma}, \quad \Gamma := \partial\Omega\times(0,T), \ \ \varphi\big|_{\partial D_T} = \frac{\partial\varphi}{\partial\nu}\Big|_{\partial D_T}=0.$$

Introduce the Hilbert space $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{\overset{\circ}{W}{}^1_{2,\Box}(D_T)} = \int\limits_{D_T} \Big[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}\Big)^2 + (\Box\, u)^2\Big]\, dx\, dt \tag{4}$$

of the classical space $\overset{\circ}{C}{}^2(\overline{D}_T, \partial D_T)$. It follows from (4) that if $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$, then $u \in \overset{\circ}{W}{}^1_2(D_T)$ and $\Box\, u \in L_2(D_T)$. Here $W^1_2(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$ and $\overset{\circ}{W}{}^1_2(D_T) = \{u \in W^1_2(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the trace theory.

Below, on the nonlinear vector function $f = (f_1, \dots, f_N)$ from (1) we impose the following requirement

$$f \in C(\mathbb{R}^n), \;\; |f(u)| \le M_1 + M_2|u|^\alpha, \;\; u \in \mathbb{R}^n, \tag{5}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^n$ and $M_i = const \ge 0$, $i = 1, 2$, and

$$0 \le \alpha = const < \frac{n+1}{n-1}\,. \tag{6}$$

Remark 1. The embedding operator $I : W^1_2(D_T) \to L_q(D_T)$ is a linear continuous compact operator for $1 < q < \frac{2(n+1)}{n-1}$ and $n > 1$. At the same time, the Nemytsky operator $K : L_q(D_T) \to L_2(D_T)$, acting according to the formula $K(u) = f(u)$, where $u = (u_1, \dots, u_N) \in L_2(D_T)$ and the vector function $f = (f_1, \dots, f_N)$ satisfies the condition (5), is continuous and bounded if $q \ge 2\alpha$. Therefore, if $\alpha < \frac{n+1}{n-1}$, then there exists a number q such that $1 < q < \frac{2(n+1)}{n-1}$ and $q \ge 2\alpha$. Thus, in this case the operator

$$K_0 = KI : W^1_2(D_T) \to L_2(D_T)$$

is continuous and compact. Moreover, from $u \in W^1_2(D_T)$ it follows that $f(u) \in L_2(D_T)$ and, if $u^m \to u$ in the space $W^1_2(D_T)$, then $f(u^m) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let the vector function f satisfy the conditions (5) and (6), $F \in L_2(D_T)$. The vector function $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ is said to be a weak generalized solution of the problem (1), (2), if for any vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ the integral equality (3) is valid, i.e.

$$\int\limits_{D_T} \Box\, u\, \Box\, \varphi\, dx\, dt + \int\limits_{D_T} f(u)\varphi\, dx\, dt = \int\limits_{D_T} F\varphi\, dx\, dt \;\; \forall\, \varphi \in \overset{\circ}{W}{}^1_{2,\Box}(D_T). \tag{7}$$

Notice that in view of Remark 1 the integral $\int\limits_{D_T} f(u)\varphi\; dx\, dt$ in the equality (7) is defined correctly, since from $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$ it follows $f(u) \in L_2(D_T)$ and, therefore, $f(u)\varphi \in L_1(D_T)$.

It is not difficult to verify that if the solution u of the problem (1), (2) belongs to the class $\overset{\circ}{C}{}^4(\overline{D}_T, \partial D_T)$ in the sense of Definition 1, then it will also be a classical solution of this problem.

Consider the following condition

$$\lim_{|u|\to\infty} \inf \frac{uf(u)}{||u||^2_{\mathbb{R}^N}} \ge 0. \tag{8}$$

Theorem 1. *Let the conditions* (5), (6) *and* (8) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1), (2) *has at least one weak generalized solution* $u \in \overset{\circ}{W}{}^1_{2,\Box}(D_T)$.

Remark 2. If the conditions of Theorem 1 are fulfilled and the Nemytsky operator $K(u)=f(u):\mathbb{R}^N\to\mathbb{R}^N$ is monotonic, i.e.

$$(K(u)-K(v))\cdot(u-v)\geq 0\ \ \forall\, u,v\in\mathbb{R}^N, \tag{9}$$

then there will hold the uniqueness of the solution of this problem.

Thus, the following theorem is valid.

Theorem 2. *Let the conditions* (5), (6) *and* (8), (9) *be fulfilled. Then for any* $F\in L_2(D_T)$ *the problem* (1), (2) *has a unique weak generalized solution in the space* $\overset{\circ}{W}{}^1_{2,\Box}(D_T)$*.*

Remark 3. The condition (9) will be fulfilled if $f\in C^1(\mathbb{R}^N)$ and the matrix $A=(\frac{\partial f_i}{\partial u_j})_{i,j=1}^N$ is defined non-negatively, i.e.

$$\sum_{i,j=1}^N \frac{\partial f_i}{\partial u_j}\,(u)\xi_i\xi_j\geq 0\ \ \forall\,\xi=(\xi_1,\dots,\xi_N),\ \ u=(u_1,\dots,u_N)\in\mathbb{R}^N.$$

As the examples show, if the conditions imposed on the nonlinear vector function f are violated, then the problem (1), (2) may not have a solution. For example, if

$$f_i(u_1,\dots,u_N)=\sum_{j=1}^N a_{ij}|u_j|^{\beta_{ij}}+b_i,\ \ i=1,\dots,N,$$

where constant numbers a_{ij},β_{ij} and b_i satisfy inequalities

$$a_{ij}>0,\ \ 1<\beta_{ij}<\frac{n+1}{n-1}\,,\ \ \sum_{i=1}^N b_i>0,$$

then the condition (8) will be violated and the problem (1), (2) will not have a solution $u\in\overset{\circ}{W}{}^1_{2,\Box}(D_T)$ for $F=\mu F^o$, where $F^o=(F_1^o,\dots,F_N^o)\in L_2(D_T)$, $G=\sum\limits_{i=1}^N F_i^o\leq 0$; $\|G\|_{L_2(D_T)}\neq 0$ for $\mu>\mu_0=\mu_0(G,\beta_{ij})=const>0$.