About the Relation Between Qualitative Properties of the Solutions of Differential Equations and Equations on Time Scales

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Differential equations on time scales were introduced by S. Hilger in [9]. His approach gave a possibility to unified theory for both discrete and continuous analysis. The theory was thoroughly stated in [1, 2]. The behavior of the solutions of the dynamic equations, defined on a family of time scales $\mathbb{T}_\lambda$ when graininess function $\mu_\lambda \to 0$ as $\lambda \to 0$ is of interest for study. In this case intervals of the time scale $[t_0, t_1]_\lambda = [t_0, t_1] \cap \mathbb{T}_\lambda$ approach $[t_0, t_1]$ (e.g. in the Hausdorff metric). The question arises about relation between properties of the solutions of equations on time scales and the solutions of boundary equations which are ordinary differential ones. It is obviously, on the finite time intervals it is not complicated to establish the convergence of solutions of dynamical equations to the corresponding solutions of differential equations. However, in case of infinite intervals this problem is not trivial.

This work is devoted to the study of existence of a bounded solution of the differential equation, defined on a family of time scales $\mathbb{T}_\lambda$ provided the graininess function μ_λ converges to zero as $\lambda \to 0$. This work extends the results of [12] about the relation between the existence of bounded solutions of differential equations and the corresponding difference equations to the case of general time scales. The main difficulty here is to obtain estimation between the solutions of differential equation and its analog for the time scale for any $\mathbb{T}_\lambda$. This makes this analysis significantly different from [12], where only special case for $\mathbb{T} = \mathbb{Z}$ was obtained.

Note, the question about existence of the two-sided solutions for dynamical equations on time scales is not trivial by itself. In comparison with the classic theorem about existence of solutions of the system of ordinary differential equations, where local both sides existence with respect to initial point holds, for the equations on time scales it is more complicated. To extend the solution to the left it is necessary the very strong regression condition holds [5]. Here we got the existence of the two-sided global bounded solution without using regression condition.

The proof of the theorems requires continuous dependence of the solutions on initial data uniformly over all time scales. It does not influence, for example, from [10], where investigation was on the fixed scale. This question is not trivial due to the topological complexity of the time scale.

The relation between properties also of the solutions of the system of ordinary differential equation and the solutions of equations on Eulerian time scales was studied before.

The paper [3] showed that the solutions of differential and the corresponding difference equations have the same oscillatory properties. The relation between stability and attractors of differential and difference equations was studied in [7]. The relation between optimal control of the systems of ordinary differential equations and dynamical equations on time scales considered in [4, 6, 11].

Let present few concepts from the monograph [1], which are used here.

Time scale $\mathbb{T}$ is an arbitrary, non-empty, closed subset of the real axis. For every $A \subset \mathbb{R}$, we denote $A_{\mathbb{T}} = A \cap \mathbb{T}$.

Define the forward and backward jump operators as $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} :\ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} :\ s < t\}$ (supplemented by $\inf \varnothing := \sup \mathbb{T}$ and $\sup \varnothing := \inf \mathbb{T}$).

The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense (LD) (left-scattered (LS), right-dense (RD) or right-scattered (RS)) if $\rho(t) = t$ ($\rho(t) < t$, $\sigma(t) = t$ or $\sigma(t) > t$) hold. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^k = \mathbb{T} \setminus \{M\}$; otherwise, we set $\mathbb{T}^k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}^d$ is said to be Δ-differentiable at $t \in \mathbb{T}^k$ if the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

exists in $\mathbb{R}^d$.

Consider the following system of differential equations

$$\frac{dx}{dt} = X(t, x), \tag{1}$$

$t \in \mathbb{R}$, $x \in D$, D is a domain in $\mathbb{R}^d$.

Consider the set of time scales $\mathbb{T}_\lambda$ and system (1) defined on $\mathbb{T}_\lambda$

$$x_\lambda^{\Delta}(t) = X(t, x_\lambda), \tag{2}$$

where $t \in \mathbb{T}_\lambda$, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^d$, $x_\lambda^{\Delta}(t)$ be delta derivative x^{Δ} for a function $x(t)$ defined on $\mathbb{T}_\lambda$, $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, $\lambda \in \Lambda \subset \mathbb{R}$, and $\lambda = 0$ is a limit point of Λ.

Assume that the function $X(t, x)$ is continuously differentiable and bounded together with its partial derivatives, i.e. $\exists\, C > 0$ such that

$$|X(t, x)| + \Big|\frac{\partial X(t, x)}{\partial t}\Big| + \Big\|\frac{\partial X(t, x)}{\partial x}\Big\| \le C \tag{3}$$

for $t \in \mathbb{R}$, $x \in D$, where $\frac{\partial X}{\partial x}$ is the corresponding Jacobi matrix.

Let $\mu_\lambda = \sup\limits_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where the graininess function $\mu_\lambda \colon \mathbb{T}_\lambda \to [0, \infty)$. Obviously, if $\mu_\lambda(t) \to 0$ when $\lambda \to 0$, then $\mathbb{T}_\lambda$ coincide (for example, in the Hausdorff metric) to a continuous time scale $\mathbb{T}_0 = \mathbb{R}$ (according classification [8]).

The following theorem holds.

Theorem 1. *Let system* (1) *has a bounded on* $\mathbb{R}$*, asymptotically stable uniformly in* $t_0 \in \mathbb{R}$ *solution* $x(t)$*, which lies in the domain* D *with some* ρ*-neighborhood. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda < \lambda_0$ *system* (2) *has a bounded on* $\mathbb{T}_\lambda$ *solution* $x_\lambda(t)$*.*

Theorem 2. *If there exists* $\lambda_0 > 0$ *such that for all* $\lambda < \lambda_0$ *system* (2) *has an asymptotically stable uniformly in* $t_0 \in \mathbb{T}_\lambda$ *and* λ *bounded on axis solution* $x_\lambda(t)$*, which lies in the domain* D *with some* ρ*-neighborhood, then system* (1) *has a bounded on axis solution.*

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