Stability of Solutions of Semidiscrete Stochastic Systems Using N. V. Azbelev's W-Method

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Semidiscrete systems of equations constitute an important subclass of so-called "hybrid systems" characterized by the presence of two components in the state space: discrete and continuous. Intuitively, this means that the dynamics is mostly continuous, but at certain instants is exposed to abrupt influences. Such systems naturally appear in applications, for example, in biological and ecological models [10, 12] as well as in the control theory [11]. Some models with impulsive actions [9] are also an important example of semidiscrete problems.

Finally, accounting for stochastic effects is an important part of any realistic approach to modeling. For example, in the population dynamics, demographic and ecological stochasticity arises due to a change in time of factors external to the system, but affecting the survival of the population, and in control theory, random coefficients can simulate, for example, inaccuracies in measurements. Therefore, the study of hybrid stochastic systems has recently attracted the attention of many specialists (see e.g. [7] and the references therein).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space.

To describe semidiscrete systems, we fix a natural number l $(1 \leq l < n)$, for which $x_1(t), \dots, x_l(t)$ $(t \geq 0)$ will be the continuous components of the state vector of the system, while $x_{l+1}(s), \dots, x_n(s)$ $(s \in N_+ \equiv \{0, 1, 2, \dots\})$ will be its discrete components. In the vector notation it will look as follows:

$$\widehat{x}(t) = \operatorname{col}(x_1(t), \dots, x_l(t)) \;\; (t \geq 0), \quad \widetilde{x}(s) = \operatorname{col}(x_{l+1}(s), \dots, x_n(s)) \;\; (s \in N_+),$$
$$x(t) = \operatorname{col}\big(x_1(t), \dots, x_l(t), x_{l+1}([t]), \dots, x_n([t])\big) \;\; (t \geq 0),$$

where $[t]$ is the integer part of the number t.

We study the moment exponential stability of solutions of the following system of linear differential and difference Itô equations with aftereffect:

$$\begin{gathered} d\widehat{x}(t) = -\sum_{j=1}^{m_1} A_{1j}(t)x(h_{1j}(t))\,dt + \sum_{i=2}^{m}\sum_{j=1}^{m_i} A_{ij}(t)x(h_{ij}(t))\,d\mathcal{B}_i(t) \;\; (t \geq 0), \\ \widetilde{x}(s+1) = \widetilde{x}(s) - \sum_{j=-\infty}^{s} A_1(s,j)x(j)h \\ + \sum_{i=2}^{m}\sum_{j=-\infty}^{s} A_i(s,j)x(j)\big(\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)\big) \;\; (s \in N_+) \end{gathered} \tag{0.1}$$

with respect to the initial conditions

$$x(\varsigma) = \varphi(\varsigma) \ \ (\varsigma < 0), \tag{0.1a}$$

$$x(0) = b. \tag{0.1b}$$

Here

- $x(t) = \operatorname{col}(x_1(t), \dots, x_l(t), x_{l+1}([t]), \dots, x_n([t]))$ $(t \geq 0)$ is a n-dimensional unknown stochastic process;
- $A_{ij}(t)$ are $l \times n$ - matrices $(i = 1, \dots, m,\ j = 1, \dots, m_i)$, where the entries of the matrices $A_{1j}(t),\ j = 1, \dots, m_1$ are progressively measurable scalar stochastic processes on interval $[0, \infty)$ with almost surely (a.s.) locally integrable trajectories, and the entries of the matrices $A_{ij}(t),\ i = 2, \dots, m,\ j = 1, \dots, m_i$ are progressively measurable scalar stochastic processes on $[0, \infty)$, whose trajectories a.s. locally square-integrable;
- $h_{ij}(t)$, $i = 1, \dots, m$, $j = 1, \dots, m_i$ are Borel measurable functions defined on $[0, \infty)$ and such that $h_{ij}(t) \leq t$ $(t \geq 0)$ are almost everywhere Lebesgue measurable for all $i = 1, \dots, m$, $j = 1, \dots, m_i$;
- h is some positive real number;
- $A_i(s, j)$ – $(n - l) \times n$ are matrices whose entries are $\mathcal{F}_s$-measurable scalar random variables for all $i = 1, \dots, m$, $s \in N_+$, $j = -\infty, \dots, s$;
- $\varphi(\varsigma) = \operatorname{col}(\varphi_1(\varsigma), \dots, \varphi_l(\varsigma), \varphi_{l+1}([\varsigma]), \dots, \varphi_n([\varsigma]))$ $(\varsigma < 0)$ is a $\mathcal{F}_0$-measurable, n-dimensional stochastic process with a.s. essentially bounded trajectories;
- $b = \operatorname{col}(b_1, \dots, b_n)$ is a $\mathcal{F}_0$-measurable n-dimensional random variable.

Under these assumptions, the problem (0.1)–(0.1b) has a unique global solution.

The moment exponential stability is defined in

Definition 0.1. System (0.1) is called exponentially q-stable with respect to the initial data if there are positive numbers c, λ such that all solutions $x(t, b, \varphi)$ $(t \in (-\infty, \infty))$ of the initial value problem (0.1), (0.1a), (0.1b) satisfy the estimate

$$\big(E|x(t,b,\varphi)|^q\big)^{1/q} \leq c \exp\{-\lambda t\} \Big(\big(E|b|^q\big)^{1/q} + \operatorname{ess\,sup}_{\varsigma<0} \big(E|\varphi(\varsigma)|^q\big)^{1/q} \Big) \ \ (t \geq 0).$$

The next definition is used in the main result of the paper.

Definition 0.2. An invertible matrix $B = (b_{ij})_{i,j=1}^m$ is called positive invertible if all elements of the matrix B^{-1} are positive.

According to [3], the matrix B will be positive invertible if $b_{ij} \leq 0$ for $i, j = 1, \dots, m$, $i \neq j$ and all diagonal minors of the matrix B are positive. In particular, matrices with strict diagonal dominance and non-positive off-diagonal elements are positive invertible.

1 Sufficient stability conditions

In this section we use a special constant c_p, which is defined in

Lemma. *For any scalar, progressive measurable stochastic process* $f(\varsigma)$

$$\bigg(E\bigg|\int\limits_0^t f(\varsigma)\,d\mathcal{B}(\varsigma)\bigg|^{2p}\bigg)^{\frac{1}{2p}} \le c_p\bigg(E\bigg(\int\limits_0^t |f(\varsigma)|^2 d\varsigma\bigg)^p\bigg)^{\frac{1}{2p}}, \tag{1.1}$$

where c_p *is some number depending on* $p \ge 1$*. Here* $\mathcal{B}(\varsigma)$ *is the scalar Wiener process.*

Estimate (1.1) follows from the inequality given in the monograph [8, p. 65], where the formulas for c_p can also be found.

Let μ be the Lebesgue measure on $[0,\infty)$. Consider three groups of conditions on the coefficients of System (0.1).

Assume that

- there exist numbers $\tau_{ij} \ge 0$, $i = 1,\dots,m$, $j = 1,\dots,m_i$ such that $0 \le t - h_{ij}(t) \le \tau_{ij}$ $(t \ge 0)$ μ-almost everywhere for all these indices;
- there exist numbers $\overline{a}_{kr}^{ij} \ge 0$, $i = 1,\dots,m$, $j = 1,\dots,m_i$, $k = 1,\dots,l$, $r = 1,\dots,n$ such that $|a_{kr}^{ij}(t)| \le \overline{a}_{kr}^{ij}$ $(t \ge 0)$ $P \times \mu$-almost everywhere for all these indices.

In addition, assume that there exist $\lambda_k \ge 0$, $k = 1,\dots,n$, for which

- the diagonal entries of the matrices $A_1(s,s)$ $(s \in N_+)$ can be represented as $a_{kk}^1(s,s) + \lambda_k$ $(s \in N_+)$, $k = l+1,\dots,n$;
- $\sum\limits_{j\in I_k} a_{kk}^{1j}(t) \ge \lambda_k$ $(t \ge 0)$ $P \times \mu$-almost everywhere $(k = 1,\dots,l)$ and some subsets $I_k \subset \{1,\dots,m_1\}$, $k = 1,\dots,l$;
- $0 < \lambda_k h < 1$ if $k = l+1,\dots,n$.

Finally, assume that there exist numbers $d_i \in N_+$, $i = 1,\dots,m$, for which

- the entries of the matrices $A_i(s,j)$ are equal to 0 P-almost everywhere for all $s \in N_+$, $j = -\infty,\dots,s-d_i-1$, $i = 1,\dots,m$;
- $|a_{kr}^i(s,j)| \le \overline{a}_{kr}^i(s,j)$ P-almost everywhere for all $i = 1,\dots,m$, $k = l+1,\dots,n$, $r = 1,\dots,n$, $s \in N_+$, $j = s-d_i,\dots,s$, and, in addition,

  $$\sup_{\tau\in N_+} \sum_{j=\nu_i(\tau)}^{\tau} \overline{a}_{kr}^1(\tau,j) < \infty \ \text{ for all } \ i = 1,\dots,m, \ \ k = l+1,\dots,n, \ \ r = 1,\dots,n,$$

  where $\nu_i(\tau) = 0$ if $0 \le \tau \le d_i$ and $\nu_i(\tau) = \tau - d_i$ if $\tau > d_i$.

The entries of the $n \times n$-matrix C are defined by

$$c_{kk} = \frac{1}{\lambda_k}\bigg(\sum_{j\in I_k} \overline{a}_{kk}^{1j}\Big(\sum_{\nu=1}^{m_1} \overline{a}_{kk}^{1\nu}\tau_{1j} + c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i} \overline{a}_{kr}^{i\nu}\sqrt{\tau_{1j}}\Big) + \sum_{j=1,\,j\in\{1,\dots,m_1\}/I_k}^{m_1} \overline{a}_{kk}^{1j}\bigg)$$
$$+\frac{c_p}{\sqrt{2\lambda_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i} \overline{a}_{kk}^{ij}, \ \ k = 1,\dots,l,$$

$$c_{kr} = \frac{1}{\lambda_k}\bigg(\sum_{j\in I_k} \overline{a}_{kr}^{1j}\Big(\sum_{\nu=1}^{m_1} \overline{a}_{kr}^{1\nu}\tau_{1j} + c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i} \overline{a}_{kr}^{i\nu}\sqrt{\tau_{1j}}\Big) + \sum_{j=1}^{m_1} \overline{a}_{kr}^{1j}\bigg)$$

$$+\frac{c_p}{\sqrt{2\lambda_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}\overline{a}_{kr}^{ij},\ \ k=1,\dots,l,\ \ r=1,\dots,n,\ \ k\neq r,$$

$$c_{kr}=\frac{1}{\lambda_k h}\bigg(h\sup_{\tau\in N_+}\sum_{j=\nu_1(\tau)}^{\tau}\overline{a}_{kr}^{1}(\tau,j)+c_p\sqrt{h}\sum_{i=2}^{m}\sup_{\tau\in N_+}\sum_{j=\nu_i(\tau)}^{\tau}\overline{a}_{kr}^{i}(\tau,j)\bigg),$$
$$k=1,\dots,l,\ \ r=1,\dots,n.$$

The above assumptions enable us to formulate the main result of this paper.

Theorem. *If the matrix $\overline{E}-C$ is positive invertible, then System* (0.1) *is exponentially $2p$-stable with respect to the initial data, i.e. in the sense of Definition 0.1. Moreover, the exponential decay rate λ of all solutions can be estimated as*

$$0<\lambda<\min\big\{\lambda_i,\ i=1,\dots,l;\ -\ln(1-\lambda_i h),\ i=l+1,\dots,n\big\}. \tag{1.2}$$

The proof of the theorem is based on the regularization method, also known as a method of model (auxiliary) equations or "N. V. Azbelev's W-method", see the monographs [1, 2] and the references therein. This approach has proven to be efficient in the theory of stochastic differential [4] and difference [5] equations. The main idea of the method is to replace functionals on the space of trajectories of solutions by the so-called "model" equation that already has the necessary property of stability and which is used to regularize the initial equation. Checking stability of the latter amounts, then, to estimating the norm of a certain integral operator or checking a positive invertibility of some matrix. The latter version of the W-method was developed in [6].

2 An example

Consider a semidiscrete system of stochastic equations with constant coefficients and bounded delays of the form:

$$\begin{gathered}
d\widehat{x}(t)=-\sum_{j=1}^{m_1}A_{1j}x(t-h_{1j})\,dt+\sum_{i=2}^{m}\sum_{j=1}^{m_i}A_{ij}x(t-h_{ij})d\mathcal{B}_i(t)\ \ (t\geq 0),\\
\widetilde{x}(s+1)=\widetilde{x}(s)-A_1\sum_{j=s-d_1}^{s}x(j)h+\sum_{i=2}^{m}A_i\sum_{j=s-d_i}^{s}x(j)\big(\mathcal{B}_i((s+1)h)-\mathcal{B}_i(sh)\big)\ \ (s\in N_+),
\end{gathered} \tag{2.1}$$

where $A_{ij}=(a_{kr}^{ij})_{k,r=1}^{l,n}$, $i=1,\dots,m$, $j=1,\dots,m_i$ are real $l\times n$-matrices and $A_i=(a_{kr}^{i})_{k=l+1,r=1}^{n}$, $i=1,\dots,m$ are real $(n-l)\times n$-matrices, and $h_{ij}\geq 0$, $i=1,\dots,m$, $j=1,\dots,m_i$ are real numbers, $h>0$ is some (sufficiently small) real number. Put also $\sum\limits_{j=1}^{m_1}a_{kk}^{1j}=a_k$, $k=1,\dots,l$ an define the entries of the $n\times n$-matrix C as follows:

$$c_{kk}=\frac{1}{a_k}\sum_{j=1}^{m_1}|a_{kk}^{1j}|\Big(\sum_{\nu=1}^{m_1}|a_{kk}^{1\nu}|h_{1j}+c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{kr}^{i\nu}|\sqrt{h_{1j}}\Big)+\frac{c_p}{\sqrt{2a_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{kk}^{ij}|,\ \ k=1,\dots,l,$$

$$c_{kr}=\frac{1}{a_k}\bigg(\sum_{j=1}^{m_1}|a_{kr}^{1j}|\Big(\sum_{\nu=1}^{m_1}|a_{kr}^{1\nu}|h_{1j}+c_p\sum_{i=2}^{m}\sum_{\nu=1}^{m_i}|a_{kr}^{i\nu}|\sqrt{h_{1j}}\Big)+\sum_{j=1}^{m_1}|a_{kr}^{1j}|\bigg)$$
$$+\frac{c_p}{\sqrt{2a_k}}\sum_{i=2}^{m}\sum_{j=1}^{m_i}|a_{kr}^{ij}|,\ \ k=1,\dots,l,\ \ r=1,\dots,n,\ \ k\neq r,$$

$$c_{kk} = \frac{c_p(d_i+1)}{a^1_{kk}\sqrt{h}} \sum_{i=2}^{m} |a^i_{kk}|, \quad k = 1+1, \dots, l,$$

$$c_{kr} = \frac{(d_1+1)|a^1_{kr}|}{a^1_{kk}} + \frac{c_p(d_i+1)}{a^1_{kk}\sqrt{h}} \sum_{i=2}^{m} |a^i_{kr}|, \quad k = 1+1, \dots, l, \quad r = 1, \dots, n, \quad k \neq r.$$

Then from Theorem we can deduce the following

Proposition. *If* $a_k > 0$, $k = 1, \dots, l$, $a^1_{kk} > 0$, $k = l+1, \dots, n$, *and the matrix* $\overline{E} - C$ *is positively invertible, then system* (2.1) *is exponentially* $2p$*-stable with respect to the initial data.*

In particular, we obtain

Corollary. *Let* $n = 2$, $l = 1$ *in system* (2.1) *and let the entries* c_{ij}, $i, j = 1, 2$ *of the* 2×2*-matrix* C *be defined as described right before Proposition. If now* $1 - c_{11} > 0$; $(1 - c_{11})(1 - c_{22}) > c_{12}c_{21}$, *then system* (2.1) *is exponentially* $2p$*-stable with respect to the initial data.*

The corollary follows from Proposition and from the fact that under the conditions of the corollary the 2×2-matrix $\overline{E} - C$ is positive invertible, since its diagonal minors are positive.

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