On the Inequality of Characteristics in the Statement of the First Darboux Problem for Second Order Hyperbolic Systems with Non-Split Principal Parts

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On the plane of independent variables x and y we consider the general system of second-order linear homogeneous differential equations

$$Au_{xx}+Bu_{xy}+Cu_{yy}+au_x+bu_y+cu=0, \tag{1}$$

where A, B, C, a, b and c are given real $N\times N$ matrices and $u=(u_1,\dots,u_N)$ is the unknown N-dimensional real vector. We assume that $\det C\neq 0$ and $N>1$ is a natural number.

By $P(x,y;\xi,\eta)$ we denote the characteristic determinant of system (1), i.e.

$$P(x,y;\xi,\eta):=\det Q(x,y;\xi,\eta),$$

where

$$Q(x,y;\xi,\eta):=A(x,y)\xi^2+B(x,y)\xi\eta+C(x,y)\eta^2$$

and ξ, η are arbitrary real parameters.

Since $\det C\neq 0$, we have the following representation

$$P(x,y;1,\lambda)=\det C\prod_{i=1}^{l}(\lambda-\lambda_i(x,y))^{k_i},$$

$$\sum_{i=1}^{l}k_i=2N,\ \ l=l(x,y),\ \ k_i=k_i(x,y),\ \ i=1,\dots,l.$$

System (1) is said to be hyperbolic at the point (x,y) if $l>1$ and all roots $\lambda_1(x,y),\dots,\lambda_l(x,y)$ of the polynomial $P(x,y;1,\lambda)$ are real (see, e.g., [1,6]).

One can readily show that [1,6]

$$k_i(x,y)\geq N-rank\ Q(x,y;1,\lambda_i(x,y)),\ \ i=1,\dots,l.$$

The hyperbolic system (1) is said to be normally hyperbolic at the point (x,y) if

$$k_i(x,y)=N-rank\ Q(x,y;1,\lambda_i(x,y)),\ \ i=1,\dots,l.$$

In the formulation of the characteristic Goursat problem for system (1), in contrast to scalar hyperbolic equations, generally speaking as was shown in [1–4], it can be ill-posed and one should be careful. In these works there are considered linear second order hyperbolic systems, for which the corresponding homogeneous characteristic Goursat problem has infinite number of linearly independent solutions. In the works [1,3–5,7] there are considered the question of the influence of lower terms on the correctness of the statement of the Goursat characteristic problem for second order hyperbolic systems with non-split principal part. As it is investigated in [6], the Goursat and Darboux first and second type problems for normally hyperbolic systems are well-posed. For the author until this day it is not known what will happen when in the case of Darboux first problem the condition of normally hyperbolicity is violated. The presented note is devoted to this question.

Suppose in system (1) that

$$N=2,\quad A=\begin{Vmatrix}1&0\\0&1\end{Vmatrix},\quad B=\begin{Vmatrix}-1&1\\1&-1\end{Vmatrix},\quad C=\begin{Vmatrix}0&-1\\-1&0\end{Vmatrix},\quad a=b=c=0,$$

and thus consider the following system

$$\begin{cases}u_{1xx}-u_{1xy}+u_{2xy}-u_{2yy}=0,\\ u_{2xx}+u_{1xy}-u_{2xy}-u_{1yy}=0.\end{cases}\tag{2}$$

System (2) is hyperbolic, since its characteristic determinate

$$D(\lambda):=(\lambda-1)^3(\lambda+1)$$

has the real roots $\lambda=1,\ \lambda=-1$.

By D_1 (D_2) we denote the domain on the plane of independent variables x and y bounded by the characteristic $x-y=0$ ($x+y=0$), $x\geq 0$ of system (2) and by the non-characteristic $y=0$, $x\geq 0$.

The Darboux first problem: in the domain $D_1(D_2)$ find a regular solution u of system (2) under the conditions

$$u\big|_{y=x}=f_1(x)\quad \big(u\big|_{y=-x}=f_2(x)\big),\quad x\geq 0,\tag{3}$$

and

$$u\big|_{y=0}=f_3,\quad x\geq 0,\tag{4}$$

where the functions f_1, $i=1,\dots,3$ are given twice continuously differentiable functions with respect to their arguments, satisfying the matching conditions: $f_1(0)=f_2(0)=f_3(0)$.

System (2) is rewritten in the following form

$$\begin{cases}\widetilde{w}_{\xi\eta}=0,\\ \widetilde{v}_{\eta\eta}=0,\end{cases}\tag{5}$$

where

$$\xi=x+y,\quad \eta=x-y,\quad \widetilde{w}(\xi,\eta):=w\Big(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\Big),\quad \widetilde{v}(\xi,\eta):=v\Big(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\Big),$$
$$w:=u_1+u_2,\quad v:=u_1-u_2.$$

By integrating system (5), we obtain

$$\widetilde{w}(\xi,\eta)=2\varphi_1(\xi)+2\varphi_2(\eta),\quad \widetilde{v}(\xi,\eta)=2\eta\varphi_3(\xi)+2\varphi_4(\xi),$$

where φ_i, $i=1,\dots,4$ are arbitrary twice continuously differentiable functions with respect to their arguments.

Returning to the previous variables, we obtain that the general classical solution of system (2) has the following form

$$\begin{cases}u_1(x,y)=\varphi_1(x+y)+\varphi_2(x-y)+(x-y)\varphi_3(x+y)+\varphi_4(x+y),\\ u_2(x,y)=\varphi_1(x+y)+\varphi_2(x-y)-(x-y)\varphi_3(x+y)-\varphi_4(x+y).\end{cases}\tag{6}$$

Based on formulas (6), we conclude that:

1) The unique solution of problem (2)–(4) in the domain D_1 is given by the formulas

$$
\begin{aligned}
u_1(x,y) = \frac{y-x}{2(x+y)}\bigg[f_1^1\Big(\frac{x+y}{2}\Big) - f_1^2\Big(\frac{x+y}{2}\Big) - f_2^1(x+y) + f_2^2(x+y)\bigg] \\
+ f_1^1\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\bigg[f_1^1\Big(\frac{x-y}{2}\Big) + f_1^2\Big(\frac{x-y}{2}\Big) - f_2^1(x-y) - f_2^2(x-y)\bigg], \\
u_2(x,y) = \frac{x-y}{2(x+y)}\bigg[f_1^1\Big(\frac{x+y}{2}\Big) - f_1^2\Big(\frac{x+y}{2}\Big) - f_2^1(x+y) + f_2^2(x+y)\bigg] \\
+ f_1^2\Big(\frac{x+y}{2}\Big) - \frac{1}{2}\bigg[f_1^1\Big(\frac{x-y}{2}\Big) + f_1^2\Big(\frac{x-y}{2}\Big) - f_2^1(x-y) - f_2^2(x-y)\bigg].
\end{aligned}
$$

2) The corresponding to (2)–(4) homogeneous problem in the domain D_2 has infinitely many linearly independent solutions given by the formulas

$$u_1(x,y) = -y\varphi_0(x+y), \quad u_2(x,y) = y\varphi_0(x+y), \quad \varphi_0(0) = 0,$$

where φ_0 is an arbitrary twice continuously differentiable function with respect to its arguments.

3) The inhomogeneous problem (2)–(4) in the domain D_2 is not always solvable for an arbitrary right-hand side.

Remark. The question of finding the well-posed problems for system (2) is certainly of scientific interest and will be the subject of further research by the author.

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