Oscillatory Properties of Solutions of Second Order Half-Linear Differential Equations

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1 Introduction

Consider the second order half-linear differential equation

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{HL}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive, continuously differentiable functions on $[a,\infty)$, $a \geqq 0$, and $\varphi_\gamma : \mathbb{R} \to \mathbb{R}$ denotes the odd function defined by

$$\varphi_\gamma(u) = |u|^\gamma \operatorname{sgn} u = |u|^{\gamma-1}u, \quad u \in \mathbb{R}, \ \ \gamma > 0.$$

It is known that all proper solutions of (HL) are either oscillatory, in which case equation (HL) itself is called oscillatory, or else nonoscillatory, in which case (HL) itself is called nonoscillatory. Our attention will be focused on oscillatory equations of the form (HL).

Let $x(t)$ be an oscillatory solution of (HL) existing on $[a,\infty)$. We denote by $\{\sigma_k\}_{k=1}^\infty$ ($\sigma_k < \sigma_{k+1}$) the sequence of zeros of $x(t)$, and by $\{\tau_k\}_{k=1}^\infty$ ($\tau_k < \tau_{k+1}$) the sequence of points at which $x(t)$ takes on extrema (i.e. local maxima or minima). Naturally, $x(\sigma_k) = 0$ and $x'(\tau_k) = 0$ for all k. The values $|x'(\sigma_k)|$ and $|x(\tau_k)|$ are referred to as the *slope* and *amplitude*, respectively, of the k-th wave of $x(t)$. We use the following notations:

$$\mathcal{A}^*[x] = \sup_k |x(\tau_k)|, \quad \mathcal{A}_*[x] = \inf_k |x(\tau_k)|, \quad \mathcal{S}^*[x] = \sup_k |x'(\sigma_k)|, \quad \mathcal{S}_*[x] = \inf_k |x'(\sigma_k)|.$$

An oscillatory solution $x(t)$ of (HL) is bounded if $\mathcal{A}^*[x] < \infty$, and unbounded if $\mathcal{A}^*[x] = \infty$. Two cases are possible for a bounded oscillatory solution: either $\lim\limits_{k\to\infty} |x(\tau_k)| = 0$ which is equivalent to $\lim\limits_{t\to\infty} x(t) = 0$, or $\liminf\limits_{k\to\infty} |x(\tau_k)| > 0$ which amounts to $\mathcal{A}_*[x] > 0$. In the former case $x(t)$ is called a *decaying oscillatory solution*, while in the latter case $x(t)$ is called an *non-decaying oscillatory solution* of (HL).

Recently, Kusano and Yoshida [1] have shown the existence and the qualitative properties, i.e., "amplitudes" and "slopes", of oscillatory solutions $x(t)$ of the linear differential equation

$$(p(t)x')' + q(t)x = 0, \ \ t \geqq a. \tag{L}$$

The purpose of this paper is to report to the QUALITDE – 2021 that some of their results can be extended to half-linear differential equations of the form (HL).

2 Main results

Our first result concerns the estimation of $\mathcal{A}^*[x]$ and $\mathcal{A}_*[x]$.

Theorem 2.1. *Let* (HL) *be oscillatory and let $x(t)$ be a solution of it satisfying the initial condition*

$$x(a) = l, \quad x'(a) = m, \tag{2.1}$$

where l and m are any given constants such that $(l, m) \neq (0, 0)$.

(i) *Suppose that $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ q(\infty) > 0, \tag{2.2}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}}q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty) < \infty. \tag{2.3}$$

(ii) *Suppose that $p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}}q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty) > 0, \tag{2.4}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ q(\infty) < \infty. \tag{2.5}$$

(iii) *Suppose that $(p(t)^{\frac{1}{\alpha}}q(t))' \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.6}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}}q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty)^{\frac{1}{\alpha}}q(\infty) < \infty. \tag{2.7}$$

(iv) *Suppose that $(p(t)^{\frac{1}{\alpha}}q(t))' \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}}q(\infty)}\Big]^{\frac{1}{\alpha+1}} \ \ \textit{if} \ \ p(\infty)^{\frac{1}{\alpha}}q(\infty) > 0, \tag{2.8}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.9}$$

Since the constants l and m in (2.1) are arbitrary, the above inequalities (2.2)–(2.9) guarantee under the indicated conditions on $p(\infty)$ and/or $q(\infty)$ that $\mathcal{A}^*[x] < \infty$ and/or $\mathcal{A}_*[x] > 0$ for all solutions $x(t)$ of (HL). Then, $\mathcal{A}^*[x] < \infty$ gives the boundedness of $x(t)$ on $[a, \infty)$ and $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ imply the non-decaying boundedness of $x(t)$ on $[a, \infty)$.

Corollary 2.1. *Suppose that* (HL) *is oscillatory. All of its solutions are bounded on* $[a,\infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions:*

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Corollary 2.2. *Supposet that* (HL) *is oscillatory. All of its solutions are non-decaying bounded on* $[a,\infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions:*

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $p(\infty) < \infty$, $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$, $q(\infty) < \infty$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

The estimation of $\mathcal{S}^*[x]$ and $\mathcal{S}_*[x]$ are given in the following

Theorem 2.2. *Let* (HL) *be oscillatory and let* $x(t)$ *be a solution of it satisfying* (2.1).

(i) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}},$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty \text{ and } q(\infty) > 0.$$

(ii) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0 \text{ and } q(\infty) < \infty,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}.$$

(iii) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) < \infty,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} \Big\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\Big\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty.$$

(iv) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} \{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0,$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) > 0.$$

Corollary 2.3. *Let (HL) be oscillatory. If* $p(t)$ *and* $q(t)$ *are monotone functions such that* $0 < p(\infty) < \infty$ *and* $0 < q(\infty) < \infty$, *then* $\mathcal{S}^*[x] < \infty$ *and* $\mathcal{S}_*[x] > 0$ *for all solutions* $x(t)$ *of* (HL).

3 Example

Example. Consider the half-linear differential equation

$$\big((\coth(t+\tau))^{\alpha}\varphi_{\alpha}(x')\big)' + k\tanh(t+\tau)\varphi_{\alpha}(x) = 0 \tag{3.1}$$

on $[0,\infty)$, where $\tau \geqq 0$ and $k > 0$ are constants. Equation (3.1) is oscillatory since the functions $p(t) = (\coth(t+\tau))^{\alpha}$ and $q(t) = k\tanh(t+\tau)$ are not integrable on $[0,\infty)$. It is clear that $p(t)$ and $q(t)$ satisfy $p'(t) \leqq 0$, $q'(t) \geqq 0$, $(p(t)^{\frac{1}{\alpha}}q(t))' = 0$, $p(0) = (\coth\tau)^{\alpha}$, $p(\infty) = 1$, $q(0) = k\tanh\tau$ and $q(\infty) = k$, all nontrivial solutions of equation (3.1) are bounded and non-decaying by (ii) and (iii) of Corollary 2.2. As regards the estimates for upper and lower amplitudes and upper and lower slopes of solutions of (3.1), we obtain, for example,

$$\begin{aligned}
\mathcal{A}^{*}[x] &\leqq \Big[\coth\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}\,(\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{A}_{*}[x] &\geqq \Big[\tanh\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}\,(\coth\tau)^{\alpha}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem 2.1, and

$$\begin{aligned}
\mathcal{S}^{*}[x] &\leqq \Big[\frac{k}{\alpha}\,\coth\tau|l|^{\alpha+1} + (\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{S}_{*}[x] &\geqq \Big[\frac{k}{\alpha}\,(\tanh\tau)^{\alpha+1}|l|^{\alpha+1} + |m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem 2.2. If in particular $\tau = 0$ and $k = \alpha$, then the upper and lower amplitudes and slopes coincide, that is,

$$\mathcal{A}^{*}[x] = \mathcal{A}_{*}[x] = \mathcal{S}^{*}[x] = \mathcal{S}_{*}[x] = \big[|l|^{\alpha+1} + |m|^{\alpha+1}\big]^{\frac{1}{\alpha+1}}.$$

This value may well be called the amplitude $\mathcal{A}[x]$ and the slope $\mathcal{S}[x]$ of the solution $x(t)$ of the equation

$$\big((\coth t)^{\alpha}\varphi_{\alpha}(x')\big)' + \alpha\tanh t\,\varphi_{\alpha}(x) = 0. \tag{3.2}$$

Notice that (3.2) is reduced to the generalized harmonic oscillator

$$(\varphi_{\alpha}(\dot z))^{\cdot} + \alpha\varphi_{\alpha}(z) = 0, \quad \cdot = \frac{d}{d\sigma}\,, \tag{3.3}$$

by means of the change of variables $(t,x) \to (\sigma,z)$ given by $\sigma = \log(\cosh t)$, $z(\sigma) = x(t)$. Equation (3.3) is known as a differential equation generating a generalized trigonometic function. Its solution $z(\sigma)$ determined by the initial condition $z(0) = 0$, $\dot z(0) = 1$ is the generalized sine function $z = S(\sigma)$ which exists on $\mathbb{R}$, is periodic with period $2\pi_{\alpha}$, $\pi_{\alpha} = \frac{2\pi}{\alpha+1}/\sin(\frac{\pi}{\alpha+1})$, and vanishes at $\sigma = k\pi_{\alpha}$, $k \in \mathbb{Z}$. It follows that (3.2) has an oscillatory solution $x(t) = S(\log(\cosh t))$ on $[0,\infty)$ whose zeros are located at $t_n = \cosh^{-1}(e^{n\pi_{\alpha}})$, $n = 0,1,2,\dots$, and whose amplitude and slope are given by $\mathcal{A}[x] = 1$ and $\mathcal{S}[x] = 1$, respectively.

References

[1] T. Kusano and N. Yoshida, Existence and qualitative behavior of oscillatory solutions of second order linear ordinary differential equations. *Acta Math. Univ. Comenian. (N.S.)* **86** (2017), no. 1, 23–50.