On One System of Nonlinear Degenerate Integro-Differential Equations of Parabolic Type

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The integro-differential equations are applied in many branches of science, such as physics, engineering, biochemistry, etc. A lot of scientific works are dedicated to the investigation and numerical resolution of integro-differential models (see, for example, [2, 7, 11, 13, 16, 18] and the references therein).

One type of nonlinear integro-differential parabolic model is obtained at the mathematical simulation of processes of electromagnetic field penetration into a substance. Based on Maxwell system [14], the mentioned model at first appeared in [3]. The integro-differential system obtained in [8] describes many other processes as well (see, for example, [7, 11] and the references therein). Equations and systems of such types still yield to the investigation for special cases. In this direction the latest and rather complete bibliography can be found in the following monographs [7, 11].

The purpose of this note is to analyze degenerate one-dimensional case of such type models. Unique solvability and convergence of the constructed semi-discrete scheme with respect to the spatial derivative and fully discrete finite difference scheme are studied.

The investigated problem has the following form. In the rectangle $Q = (0,1)\times(0,T]$, where T is a fixed positive constant, we consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left\{\left[\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau+\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]\frac{\partial U}{\partial x}\right\}=f(x,t), \tag{1}$$

$$\frac{\partial V}{\partial t}-\frac{\partial}{\partial x}\left\{\left[\int_0^t\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]d\tau+\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]\frac{\partial V}{\partial x}\right\}=g(x,t), \tag{2}$$

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0,\ \ t\in[0,T], \tag{3}$$

$$U(x,0)=U_0(x),\quad V(x,0)=V_0(x),\ \ x\in[0,1]. \tag{4}$$

Here $f=f(x,t)$, $g=g(x,t)$, $U_0=U_0(x)$, $V_0=V_0(x)$ are given functions of their arguments and $U=U(x,t)$, $V=V(x,t)$ are unknown functions.

It is necessary to mention that (1), (2) is a degenerate type parabolic system with integro-differential and p-Laplacian ($p=4$) terms. Let us note that non-degenerate variants of (1)–(4) type problem for more general nonlinearities are studied in [6]. Many works are devoted to the investigation of multi-dimensional cases of such type equations and systems as well (see, for example, [1, 4, 7, 9–12, 15] and the references therein). We would also like to note that in recent years special attention has been paid to the construction and investigation of splitting models for this type and their generalized variants of multi-dimensional integro-differential equations (see, for example, [7, 9, 10] and the references therein).

As it was already mentioned, (1), (2) type models arise, on the one hand, when solving real applied problems, and on the other hand, as a natural generalization of some nonlinear parabolic equations and systems studied for example, in [16, 17] and in many other works as well.

Problems of (1)–(4) type at first were studied in [1], where the monotonicity of the considered operator is proved and the unique solvability is obtained.

Applying one modification of compactness method developed in [17] (see also [16]) the following uniqueness and existence statement takes place.

Theorem 1. *If $f,g\in W_2^1(Q)$, $f(x,0)=g(x,0)=0$, $U_0,V_0\in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U, V of problem* (1)–(4) *satisfying the following properties:*

$$U,V\in L_4(0,T;\overset{\circ}{W}{}_4^1(0,1)\cap W_2^2(0,1)),\ \ \frac{\partial U}{\partial t}\,,\frac{\partial V}{\partial t}\in L_2(Q),\ \ \sqrt{T-t}\,\frac{\partial^2 U}{\partial t}\,,\sqrt{T-t}\,\frac{\partial^2 V}{\partial t}\in L_2(Q).$$

Here usual well-known spaces are used.

In order to describe the space-discretization for problem (1)–(4), let us introduce nets: $\omega_h=\{x_i=ih,\ i=1,2,\dots,M-1\}$, $\overline{\omega}_h=\{x_i=ih,\ i=0,1,\dots,M\}$ with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. The semi-discrete approximation at (x_i,t) is designed by $u_i=u_i(t)$, $v_i=v_i(t)$. The exact solution of problem (1)–(4) at point (x_i,t) is denoted by $U_i=U_i(t)$, $V_i=V_i(t)$.

Approximating the space derivatives by a forward and backward differences:

$$w_{x,i}=\frac{w_{i+1}-w_i}{h}\,,\quad w_{\overline{x},i}=\frac{u_i-w_{i-1}}{h}\,,$$

let us correspond the following semi-discrete scheme to problem (1)–(4):

$$\frac{du_i}{dt}-\bigg\{\bigg[\int\limits_0^t\big[(u_{\overline{x},i})^2+(v_{\overline{x},i})^2\big]\,d\tau+(u_{\overline{x},i})^2+(v_{\overline{x},i})^2\bigg]\,u_{\overline{x},i}\bigg\}_{x,i}=f(x_i,t),\ \ i=1,\dots,M-1, \tag{5}$$

$$\frac{dv_i}{dt}-\bigg\{\bigg[\int\limits_0^t\big[(u_{\overline{x},i})^2+(v_{\overline{x},i})^2\big]\,d\tau+(u_{\overline{x},i})^2+(v_{\overline{x},i})^2\bigg]\,v_{\overline{x},i}\bigg\}_{x,i}=g(x_i,t),\ \ i=1,\dots,M-1, \tag{6}$$

$$u_0(t)=u_M(t)=v_0(t)=v_M(t)=0,\ \ t\in[0,T], \tag{7}$$

$$u_i(0)=U_{0,i},\ \ i=0,1,\dots,M, \tag{8}$$

which approximates problem (1)–(4) on smooth solutions with the first order of accuracy with respect to spatial step h.

The semi-discrete scheme (5)–(8) represents a Cauchy problem for nonlinear system of ordinary integro-differential equations. It is stable with respect to initial data and right-hand side of equations (5), (6) in the norm

$$\|w\|_h=(w,w)_h^{1/2},\quad (w,z)_h=\sum_{i=1}^{M-1}w_iz_ih.$$

It is not difficult to obtain the following estimate for (5)–(8)

$$\|u\|_h^2+\|v\|_h^2+\int\limits_0^t\big[\,||u_{\overline{x}}||_h^2+||v_{\overline{x}}||_h^2\,\big]\,d\tau<C,$$

where the norm under the integral is defined as follows

$$||w]|_h^2 = (w,w]_h = \sum_{i=1}^{M} w_i w_i h.$$

Here C denotes the positive constant independent of the mesh parameter h. This estimate gives the above-mentioned stability as well as the global existence of a solution to problem (5)–(8).

In Theorems 2 and 3, using an approach of the work [5] for investigation the finite-difference scheme, the convergence of the approximate solutions are stated.

For earlier work on discretization in time or space, or both, of models such as (1), (2), see, e.g., [5–12].

The following statement takes place.

Theorem 2. *The solution*

$$u(t) = \big(u_1(t), u_2(t), \dots, u_{M-1}(t)\big), \quad v(t) = \big(v_1(t), v_2(t), \dots, v_{M-1}(t)\big)$$

of the semi-discrete scheme (5)–(8) *converges to the solution of problem* (1)–(4)

$$U(t) = \big(U_1(t), U_2(t), \dots, U_{M-1}(t)\big), \quad V(t) = \big(V_1(t), V_2(t), \dots, V_{M-1}(t)\big)$$

in the norm $\|\cdot\|_h$ *as* $h \to 0$.

In order to describe the fully discrete analog of problem (1)–(4), let us construct grid on the rectangle $\overline{Q}$. For using the time-discretization in equations (1), (2) the net is introduced as follows $\omega_\tau = \{t_j = j\tau,\ j = 0,1,\dots,J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

Let us correspond the following implicit finite difference scheme to problem (1)–(4), where the terms with time derivatives in (5), (6) are approximated using the forward finite difference formula:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{\Big[\tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big] + (u_{\overline{x},i}^{j+1})^2 + (v_{\overline{x},i}^{j+1})^2\Big] u_{\overline{x},i}^{j+1}\bigg\}_{x,i} = f_i^{j+1}, \tag{9}$$

$$\frac{v_i^{j+1} - v_i^j}{\tau} - \bigg\{\Big[\tau \sum_{k=1}^{j+1} \big[(u_{\overline{x},i}^k)^2 + (v_{\overline{x},i}^k)^2\big] + (u_{\overline{x},i}^{j+1})^2 + (v_{\overline{x},i}^{j+1})^2\Big] v_{\overline{x},i}^{j+1}\bigg\}_{x,i} = g_i^{j+1}, \tag{10}$$

$$i = 1,2,\dots,M-1, \quad j = 0,1,\dots,J-1;$$

$$u_0^j = u_M^j = v_0^j = v_M^j = 0, \quad j = 0,1,\dots,J, \tag{11}$$

$$u_i^0 = U_{0,i}, \quad v_i^0 = V_{0,i}, \quad i = 0,1,\dots,M. \tag{12}$$

Thus, the system of nonlinear algebraic equations (9)–(12) is obtained, which approximates problem (1)–(4) on sufficiently smooth solution with the first order of accuracy with respect to time and spatial steps τ and h.

The following estimate can be obtained easily for the solution of the finite difference scheme (9)–(12)

$$\max_{0 \le j\tau \le T} \big(\|u^j\|_h^2 + \|v^j\|_h^2\big) + \sum_{k=1}^{J} \big(\, ||u_{\overline{x}}^k]|_h^2 + ||v_{\overline{x}}^k]|_h^2 \,\big)\tau < C,$$

which guarantees the stability and solvability of scheme (9)–(12). It is proved also that system (9)–(12) has a unique solution. Here C represents positive constant independent from time and spatial steps τ and h.

The following main conclusion is valid for scheme (9)–(12).

Theorem 3. *The solution*

$$u^j=\left(u_1^j,u_2^j,\dots,u_{M-1}^j\right),\quad v^j=\left(v_1^j,v_2^j,\dots,v_{M-1}^j\right),\ \ j=1,2,\dots,J$$

of the difference scheme (9)–(12) *converges to the solution*

$$U^j=\left(U_1^j,U_2^j,\dots,U_{M-1}^j\right),\quad V^j=\left(V_1^j,V_2^j,\dots,V_{M-1}^j\right),\ \ j=1,2,\dots,J$$

of problem (1)–(4) *in the norm* $\|\cdot\|_h$ *as* $\tau\to 0$ *and* $h\to 0$*.*

Note that for solving the difference scheme (9)–(12) Newton iterative process is used. Various numerical experiments are done. These experiments agree with theoretical research.

Statements such Theorems 1–3 for (1), (2) type equation are stated in [8]. As it was mentioned in [8], it is very interesting to looking for assumptions on the data of the considered problem (1)–(4) that provide the regularity for the solution $U(x,t)$, $V(x,t)$, which is required for obtaining rates of convergence in Theorems 2 and 3 as well as the optimal rates of convergence. It is important also to study more general nonlinearities for such kind degenerate and non-degenerate equations and systems.

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