Anti-Perron's Effect of Changing Linear Exponentially Decreasing Perturbations of All Positive Characteristic Exponents of the First Linear Approximation to Negative Ones

N. A. Izobov

Department of Differential Equations, Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: izobov@im.bas-net.by

A. V. Il'in

Lomonosov Moscow State University, Moscow, Russia
E-mail: iline@cs.msu.su

Consider the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq t_0, \tag{1}$$

with bounded infinitely differentiable coefficients and positive characteristic exponents $\lambda_n(A) \geq \cdots \geq \lambda_1(A) > 0$, as well as the perturbed systems

$$\dot{y} = A(t)y + Q(t)y, \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{2}$$

with infinitely differentiable exponentially decreasing perturbation $n \times n$-matrices Q satisfying the estimate

$$\|Q(t)\| \leq C_Q e^{-\sigma t}, \ \ \sigma > 0, \ \ C_Q = const, \ \ t \geq t_0. \tag{3}$$

There arises the question on the existence of such, for example, two-dimensional system (1) and perturbation (3) that the perturbed system (2) has a nontrivial solution with a negative Lyapunov exponent. The solution to this (first) problem may serve as a preliminary step in solving the more important (second) problem about the existence of nontrivial solutions with negative exponents of a nonlinear differential system

$$\dot{y} = A(t)y + f(t,y), \ \ y \in \mathbb{R}^n, \ \ t \geq t_0, \tag{4}$$

with an infinitely differentiable m-perturbation $f(t,y)$:

$$\|f(t,y)\| \leq C_f \|y\|^m, \ \ y \in \mathbb{R}^n, \ \ C_f = const, \ \ t \geq t_0$$

of $m > 1$ order of smallness in the nighbourhood of the origin $y = 0$ and admissible growth outside it in the "anti-Perron" case when all characteristic exponents of linear approximation (1) are positive. Indeed, according to the principle of linear inclusion [1, p. 159], any solution $y_0(t) \neq 0$ of system (4), infinitely extendably to the right, with a negative exponent, is likewise a solution of system (2) with exponentially decresing perturbation $Q_{y_0}(t)$, satisfying the condition

$$\|Q_{y_0}(t)\| \leq C_f \|y_0(t)\|^{m-1}, \ \ t \geq t_0.$$

Therefore, in the case of admissible negative solution of the first problem there follows the same solution of the second problem.

Note that in the Perron effect ([5], [4, pp. 50-51]) of changing the values of negative characteristic exponents of system (1) by positive exponents of solutions of system (4) we have obtained in [4] and [5] a finale complete description of sets of all positive and all negative (including those in the absence of the latter) exponents of solutions of system (4) for which all nontrivial solutions are infinitely extendable to the right and have bounded finite exponents.

The present paper is devoted to the positive solution of the first problem.

Theorem 1. *For any parameters $\lambda_2 \geq \lambda_1 > 0$, $\theta > 1$ and $\sigma \in (0, \lambda_1 + \theta^{-1}\lambda_2)$ there exist:*

1) *the two-dimensional linear system* (1) *with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, 2$;*

2) *the infinitely differentiable exponentially decreasing and satisfying estimate* (3) *perturbation $Q(t)$*

such that the perturbed linear system (2) *has a unique (among all its linear independent) solution $y(t)$ with a negative Lyapunov exponent, equal to*

$$\lambda_0 = \frac{\theta\sigma - \theta\lambda_1 - \lambda_2}{\theta - 1}.$$

There likewise arises the question on a possible number of linearly independent solutions with negative Lyapunov exponents for the n-dimensional linear perturbed system (2) in which the first approximation system (1) has all positive characteristic exponents, and the perturbation $Q(t)$ is exponentially decreasing.

The following theorem is valid.

Theorem 2. *For any parameters*

$$\lambda_n \geq \dots \geq \lambda_2 \geq \lambda_1 > 0, \;\; n \geq 3, \;\; \theta > 1, \;\; 0 < \sigma < \lambda_1 + \theta^{-1}\lambda_2$$

there exist:

1) *the n-dimensional system* (1) *with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_i(A) = \lambda_i$, $i = 1, \dots, n$;*

2) *the infinitely differentiable exponentially decreasing and satisfying estimate* (3) *perturbation $Q(t)$*

such that the n-dimensional perturbed system (2) *has exactly $n - 1$ linear independent solutions*

$$Y_1(t), \dots, Y_{n-1}(t)$$

with negative exponents

$$\lambda[Y_i] = \frac{\sigma\theta - \theta\lambda_1 - \lambda_{i+1}}{\theta - 1} \equiv \Lambda_i, \;\; i = 1, \dots, n-1.$$

Proof of Theorem 2 is based on the statement of Theorem 1 and its proof.

Remark. Is the statement:

$$\text{if } \; \lambda_i(A) > 0, \;\; i = 1, \dots, n, \;\; \text{then } \; \lambda_n(A + Q) > 0$$

valid for any piecewise continuous bounded $n \times n$-matrix $A(t)$ and exponentially decreasing $n \times n$-perturbation $Q(t)$?

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