Positive Solutions to Boundary Value Problems for Nonlinear Functional Differential Equations

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Consider a boundary value problem for a functional differential equation

$$u'(t)=\ell(u)(t)+\lambda F(u)(t) \ \text{ for a.e. } \ t\in[a,b], \quad h(u)=0. \tag{1}$$

Here, $\ell: C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ and $h: C([a,b];\mathbb{R})\to\mathbb{R}$ are linear bounded operators, $F: C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions, and $\lambda\in\mathbb{R}$ is a parameter. By a solution to the problem (1) we understand an absolutely continuous function $u:[a,b]\to\mathbb{R}$ that satisfies the equation in (1) almost everywhere on $[a,b]$ and satisfies the boundary condition in (1). We say that a solution u to (1) is positive if $u(t)>0$ for $t\in[a,b]$.

Although the assumptions of the main results do not exclude the case when $F(0)\not\equiv 0$, the main importance of our results is that they are applicable in the case when the problem (1) possesses a trivial solution, i.e., $F(0)(t)=0$ for a.e. $t\in[a,b]$.

Notation 1.

$\mathbb{N}$ is the set of all natural numbers, $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+=\,]0,+\infty[\,$, $\mathbb{R}_0^+=[0,+\infty[\,$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $v:[a,b]\to\mathbb{R}$ with the norm $\|v\|_C=\max\{|v(t)|:\ t\in[a,b]\}$.

If $D\subset\mathbb{R}$, then $C_h([a,b];D)=\{u\in C([a,b];\mathbb{R}):\ u(t)\in D \text{ for } t\in[a,b],\ h(u)=0\}$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p:[a,b]\to\mathbb{R}$ with the norm $\|p\|_L=\int\limits_a^b|p(s)|\,ds$.

If $D\subset\mathbb{R}$, then $L([a,b];D)=\{p\in L([a,b];\mathbb{R}):\ p(t)\in D \text{ for a.e. } t\in[a,b]\}$.

If $A: C([a,b];\mathbb{R})\to C([a,b];\mathbb{R})$ is a linear bounded operator, by $\|A\|$ we denote the norm of A.

Definition 1. We say that a pair of operators (ℓ,h) belongs to the set $\mathcal{V}^+$ if every nontrivial absolutely continuous function $u:[a,b]\to\mathbb{R}$, satisfying

$$u'(t)\geq\ell(u)(t) \ \text{ for a.e. } \ t\in[a,b], \quad h(u)=0, \tag{2}$$

admits the inequality

$$u(t)>0 \ \text{ for } \ t\in[a,b].$$

Definition 2. We say that a pair of operators (ℓ, h) belongs to the set $\mathcal{U}^+$ if there exists $c > 0$ such that every absolutely continuous function $u : [a, b] \to \mathbb{R}$ satisfying (2) admits the inequality

$$u(t) \geq c \int_a^b [u'(s) - \ell(u)(s)]\, ds \ \text{ for } \ t \in [a, b].$$

It can be easily seen that if $(\ell, h) \in \mathcal{V}^+$, resp. $(\ell, h) \in \mathcal{U}^+$, then the homogeneous problem

$$u'(t) = \ell(u)(t) \ \text{ for a.e. } \ t \in [a, b], \quad h(u) = 0 \tag{3}$$

has only the trivial solution. Note also that $\mathcal{U}^+ \subseteq \mathcal{V}^+$. However, $\mathcal{U}^+ \neq \mathcal{V}^+$ in general.

Now we formulate some of the assumptions of the main results.

(H.1) F transforms $C_h([a, b]; \mathbb{R}_0^+)$ into $L([a, b]; \mathbb{R}_0^+)$ and it is not the zero operator, i.e., there exists $x_0 \in C_h([a, b]; \mathbb{R}_+)$ such that

$$\int_a^b F(x_0)(s)\, ds > 0.$$

(H.2) F is sublinear with respect to $C_h([a, b]; \mathbb{R}_0^+)$, i.e., there exists a Carathéodory function $\eta : [a, b] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ non-decreasing in the second variable such that

$$F(v)(t) \leq \eta(t, \|v\|_C) \ \text{ for a.e. } \ t \in [a, b], \ \ v \in C_h([a, b]; \mathbb{R}_0^+)$$

and

$$\lim_{x \to +\infty} \frac{1}{x} \int_a^b \eta(s, x)\, ds = 0.$$

(H.3) F is nondecreasing in the neighbourhood of zero, i.e., for every $\rho > 0$ there exists $m_\rho \in C_h([a, b]; \mathbb{R}_+)$ such that $m_\rho(t) \leq \rho$ for $t \in [a, b]$ and

$$F(y)(t) \leq F(x)(t) \ \text{ for a.e. } \ t \in [a, b]$$

whenever $x, y \in C_h([a, b]; \mathbb{R}_+)$,

$$y(t) \leq m_\rho(t) \ \text{ and } \ y(t) \leq x(t) \leq \rho \ \text{ for } \ t \in [a, b].$$

(H.4) F is concave in the neighbourhood of zero, i.e., for every $x \in C_h([a, b]; \mathbb{R}_+)$ there exists $\mu_x > 0$ such that

$$\mu F(x)(t) \leq F(\mu x)(t) \ \text{ for a.e. } \ t \in [a, b], \ \ \mu \in \,]0, \mu_x[\,.$$

Notation 2. Let $\lambda \in \mathbb{R}$. Then by $\mathcal{S}(\lambda)$ we denote the set of all positive solutions to (1) for corresponding λ.

Theorem 1. *Let* $(\ell, h) \in \mathcal{V}^+$ *and let* (H.1)–(H.4) *be fulfilled. Then there exists a critical parameter* $\lambda_c \geq 0$ *such that*

(i) *the problem* (1) *has a positive solution provided* $\lambda > \lambda_c$*;*

(ii) *the problem* (1) *has no positive solution provided* $\lambda < \lambda_c$*.*

Moreover,

$$\lim_{\lambda\to+\infty}\inf\big\{\|u\|_C:\ u\in\mathcal{S}(\lambda)\big\}=+\infty.$$

If, in addition, $(\ell,h)\in\mathcal{U}^+$, *then for every* $\rho>0$ *there exists* $\lambda(\rho)>\lambda_c$ *such that*

$$u(t)>\rho\ \ \text{for}\ \ t\in[a,b],\ \ u\in\mathcal{S}(\lambda),\ \ \lambda>\lambda(\rho).$$

As for the critical case $\lambda=\lambda_c$, the existence or nonexistence of a positive solution to (1) depends on the properties of the operator F; both cases can occur. If we slightly strengthen the assumption (H.4), in particular, if we assume

(H.4′) For every $x\in C_h([a,b];\mathbb{R}_+)$ there exists $\mu_x>0$ such that

$$\mu F(x)(t)\le F(\mu x)(t)\ \text{ for a.e. }\ t\in[a,b],\ \ \mu\in\,]0,\mu_x[$$

and

$$\mu_0\int_a^b F(x)(s)\,ds<\int_a^b F(\mu_0x)(s)\,ds$$

for some $\mu_0\in\,]0,\mu_x[$

instead, we can establish a result about the nonexistence of a positive solution to (1) with $\lambda=\lambda_c$.

Theorem 2. *Let* $(\ell,h)\in\mathcal{V}^+$ *and let* (H.1)–(H.3)*, and* (H.4′) *be fulfilled. Then* $\mathcal{S}(\lambda_c)=\varnothing$ *and*

$$\lim_{\lambda\to\lambda_c^+}\sup\big\{\|u\|_C:\ u\in\mathcal{S}(\lambda)\big\}=0.$$

Suppose that the operator F includes a linear part, i.e.,

$$F(v)(t)=\widetilde{F}(v,v)(t)\ \text{ for a.e. }\ t\in[a,b],\ \ v\in C([a,b];\mathbb{R}),$$

where $\widetilde{F}:C([a,b];\mathbb{R})\times C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is a continuous operator satisfying the Carathéodory conditions and it is linear and nondecreasing in the first variable. Therefore, instead of (1) we consider the problem

$$u'(t)=\ell(u)(t)+\lambda\widetilde{F}(u,u)(t)\ \text{ for a.e. }\ t\in[a,b],\quad h(u)=0,\tag{4}$$

where ℓ and λ are the same as in (1) and $\widetilde{F}$ is described above. The set of all positive solutions to (4) we denote again by $\mathcal{S}(\lambda)$ as (4) is a particular case of (1).

Theorem 3. *Let* $(\ell,h)\in\mathcal{V}^+$ *and let* (H.1)–(H.3) *and* (H.4′) *be fulfilled. Then* $\lambda_c>0$*, the problem*

$$u'(t)=\ell(u)(t)+\lambda_c\widetilde{F}(u,0)(t)\ \textit{ for a.e. }\ t\in[a,b],\quad h(u)=0\tag{5}$$

has a positive solution u_c*, the set of solutions to* (5) *is one-dimensional (generated by* u_c*), and*

$$(T_\lambda,h)\in\mathcal{V}^+,\ \ \textit{resp.}\ \ (T_\lambda,h)\in\mathcal{U}^+,\ \ \textit{for}\ \ \lambda\in\,]0,\lambda_c[\,,$$

where

$$T_\lambda(v)(t)\overset{def}{=}\ell(v)(t)+\lambda\widetilde{F}(v,0)(t)\ \textit{ for a.e. }\ t\in[a,b],\ \ v\in C([a,b];\mathbb{R}),$$

provided $(\ell,h)\in\mathcal{V}^+$*, resp.* $(\ell,h)\in\mathcal{U}^+$.

Theorem 3 gives us a method how to calculate the precise value of λ_c in the cases where F includes a linear part. Indeed, define an operator $A : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ by

$$A(x)(t) \stackrel{def}{=} \int_a^b G(t,s)\widetilde{F}(x,0)(s)\,ds \ \text{ for } \ t \in [a,b], \ \ x \in C([a,b];\mathbb{R}),$$

where G is Green's function to (3). Then

$$u_c(t) = \lambda_c A(u_c)(t) \ \text{ for } \ t \in [a,b],$$

i.e., $1/\lambda_c$ is the first eigenvalue to A corresponding to the positive eigenfunction u_c. Therefore, according to Krasnoselski's theory and Gelfand's formula,

$$\lambda_c = \lim_{n\to+\infty} \frac{1}{\sqrt[n]{\|A^n\|}}\,.$$

Application

Most of population models with a delayed harvesting term can be represented as an equation

$$u'(t) = -\delta(t)u(t) - H(t)u(t-\sigma(t)) + \lambda \sum_{k=1}^{N} P_k(t)u(t-\tau_k(t))f_k(u(t-\tau_k(t))), \tag{6}$$

where $N \in \mathbb{N}$,

(A.1) (i) $\delta, H, P_k : \mathbb{R} \to \mathbb{R}_0^+$ $(k=1,\dots,N)$ are T-periodic locally integrable functions,

$$\int_0^T \big[\delta(s) + H(s)\big]\,ds > 0, \qquad \int_0^T \sum_{k=1}^{N} P_k(s)\,ds > 0,$$

(ii) $\sigma : \mathbb{R} \to [0,\sigma_*]$, $\tau_k : \mathbb{R} \to [0,\tau_*]$ $(k=1,\dots,N)$ are T-periodic locally measurable functions (σ_* and τ_* are non-negative constants),

(iii) $f_k : \mathbb{R}_0^+ \to \mathbb{R}_+$ $(k=1,\dots,N)$ are continuous decreasing functions that are continuously differentiable at some neighbourhood of zero, and

$$\lim_{x\to+\infty} f_k(x) = 0 \ \ (k=1,\dots,N).$$

By a T-periodic solution to (6) we understand a T-periodic locally absolutely continuous function defined on $\mathbb{R}$ and satisfying the equality (6) for almost every $t \in \mathbb{R}$.

Theorem 4. *Let* (A.1) *be fulfilled, and let there exist $\gamma : \mathbb{R} \to \mathbb{R}_+$ that is locally absolutely continuous such that*

$$\gamma'(t) \leq -\delta(t)\gamma(t) - H(t)\gamma(t-\sigma(t)) \ \text{ for a.e. } \ t \in \mathbb{R}. \tag{7}$$

Then $(\ell,h) \in \mathcal{U}^+$ and F satisfies (H.1)–(H.3) *and* (H.4′) *with $\mu_x = 1$ for all $x \in C([0,T];\mathbb{R}_+)$, where the operators $\ell, F : C([0,T];\mathbb{R}) \to L([0,T];\mathbb{R})$ and $h : C([0,T];\mathbb{R}) \to \mathbb{R}$ are defined by*

$$\ell(v)(t) \stackrel{def}{=} -\delta(t)v(t) - H(t)v(\sigma_0(t)), \quad h(v) \stackrel{def}{=} v(0) - v(T) \ \text{ for } \ v \in C([0,T];\mathbb{R}),$$

$$
\begin{aligned}
F(v)(t) &\overset{def}{=} \sum_{k=1}^{N} P_k(t)v(\tau_{0k}(t))f_k(v(\tau_{0k}(t))) \ \ \textit{for} \ \ v \in C([0,T];\mathbb{R}),\\
\sigma_0(t) &\overset{def}{=} t-\sigma(t) + \left\lfloor \frac{T-(t-\sigma(t))}{T} \right\rfloor T \ \ \textit{for a.e.} \ \ t \in [0,T],\\
\tau_{0k}(t) &\overset{def}{=} t-\tau_k(t) + \left\lfloor \frac{T-(t-\tau_k(t))}{T} \right\rfloor T \ \ \textit{for a.e.} \ \ t \in [0,T] \ \ (k=1,\dots,N).
\end{aligned}
$$

One of the efficient conditions guaranteeing the existence of a positive γ satisfying (7) is

$$
\int\limits_{t-\sigma(t)}^{t} H(s) \exp\bigg(\int\limits_{s-\sigma(s)}^{s} \delta(\xi)\,d\xi \bigg)\,ds \leq \frac{1}{e} \ \ \text{for a.e.} \ \ t \in [0,T].
$$

Another conditions guaranteeing the inclusion $(\ell,h) \in \mathcal{U}^+$ are

$$
\text{either} \ \ \int\limits_{0}^{T} \big[\delta(s)+H(s)\big]\,ds < 1 \ \ \text{or} \ \ \int\limits_{0}^{T} H(s) \exp\bigg(\int\limits_{s-\sigma(s)}^{s} \delta(\xi)\,d\xi \bigg)\,ds < 1
$$

provided (A.1)(i) is fulfilled.