On Some Estimates for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \ \ x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non–negative values such that the following integral conditions hold:

$$\int\limits_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \ \ \gamma \neq 0, \tag{1.3}$$

$$\int\limits_0^1 x(1-x)Q(x)\,dx < \infty. \tag{1.4}$$

A function y is a *solution to problem* (1.1), (1.2) if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

This work gives estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \ \text{ and } \ M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

Some of these results were obtained using approaches and ideas applied in works [1, 4–6].

In Theorem 1 [3], it was proved that if condition (1.4) does not hold, then for any $0 \le p \le \infty$, there is no non-trivial solution y of equation (1.1) with the properties $y(0) = 0$, $y'(0) = p$.

From the results of [4, Chapter 1, § 2, Theorem 3] it follows that $T_{\alpha,\beta,\gamma}$ is empty provided $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, for other values α, β, γ, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty. Thus, for $\gamma < 0$, $\alpha \le 2\gamma - 1$ or $\beta \le 2\gamma - 1$, there is no function Q satisfying (1.3) and (1.4) taken together and, as a consequence, the first eigenvalue of problem (1.1), (1.2) does not exist.

Consider the functional

$$R[Q,y] = \frac{\int\limits_0^1 y'^2\,dx - \int\limits_0^1 Q(x)y^2\,dx}{\int\limits_0^1 y^2\,dx}\,.$$

If condition (1.4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$. In order to show it, let us consider the set Γ_* of functions $y \in H_0^1(0,1)$ such that

$$\int_0^1 y^2\,dx = 1$$

and the functional

$$I[Q,y] = \int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx.$$

For any $y \in H_0^1(0,1)$ and $x \in (0,1)$, by the Hölder inequality, we have

$$y^2(x) = \Big(\int_0^x y'(t)\,dt\Big)^2 \le x \int_0^x y'^2(t)\,dt,$$

$$y^2(x) = \Big(-\int_x^1 y'(t)\,dt\Big)^2 \le (1-x) \int_x^1 y'^2(t)\,dt.$$

Then

$$\frac{y^2}{x(1-x)} = \frac{y^2}{x} + \frac{y^2}{1-x} \le \int_0^x y'^2(t)\,dt + \int_x^1 y'^2(t)\,dt = \int_0^1 y'^2(t)\,dt$$

and

$$\int_0^1 Q(x)y^2 dx \le \Big(\int_0^1 y'^2\,dx\Big) \int_0^1 x(1-x)Q(x)\,dx.$$

For some positive k, consider

$$E_k = \big\{x \in [0,1] \mid \ Q(x) \le k\big\}, \quad \overline{E}_k = \big\{x \in [0,1] \mid \ Q(x) > k\big\}.$$

We have

$$\int_0^1 Q(x)y^2\,dx = \int_{E_k} Q(x)y^2\,dx + \int_{\overline{E}_k} Q(x)y^2\,dx \le k\int_0^1 y^2 dx + \int_0^1 y'^2\,dx \int_{\overline{E}_k} x(1-x)Q(x)\,dx.$$

Since the integral $\int_0^1 x(1-x)Q(x)\,dx$ is finite and the measure of $\overline{E}_k$ tends to 0 as $k \to \infty$, then $\int_{\overline{E}_k} x(1-x)Q(x)\,dx$ tends to 0 as $k \to \infty$ and we can choose $k = k_*$ so that

$$\int_{\overline{E}_{k_*}} x(1-x)Q(x)\,dx \le \frac{1}{2}\,.$$

Then

$$\int_0^1 Q(x)y^2 dx \le k_* + \frac{1}{2}\int_0^1 y'^2\,dx$$

and

$$\int_0^1 y'^2\,dx - \int_0^1 Q(x)y^2\,dx \geq \frac{1}{2}\int_0^1 y'^2\,dx - k_* \geq -k_*.$$

Thus, if condition (1.4) is satisfied, then for any $Q \in T_{\alpha,\beta,\gamma}$, $I[Q,y]$ is bounded below in Γ_*, $R[Q,y]$ is bounded below in $H_0^1(0,1)$, and

$$\inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y] = \inf_{y\in\Gamma_*} I[Q,y].$$

It was proved [3] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$M_{\alpha,\beta,\gamma} = \sup_{Q\in T_{\alpha,\beta,\gamma}} \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y] \leq \inf_{y\in H_0^1(0,1)\setminus\{0\}} \frac{\int_0^1 y'^2\,dx}{\int_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1. *If $\gamma > 1$, $\alpha,\beta < 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $m_{\alpha,\beta,\gamma} = R[Q_*,u]$. Moreover, u satisfies the equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}} \tag{2.1}$$

and the integral condition

$$\int_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{2.2}$$

Theorem 2.2.

(1) *If $\gamma = 1$, $\alpha,\beta \leqslant 0$, then $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.*

(2) *If $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ or $\alpha \leqslant 0 < \beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(3) *If $\gamma = 1$, $0 < \alpha,\beta \leqslant 1$, then $m_{\alpha,\beta,\gamma} \geqslant 0$.*

(4) *If $\gamma > 1$, $\alpha,\beta \leqslant \gamma$, then $m_{\alpha,\beta,\gamma} = 0$.*

(5) *If $\gamma \geqslant 1$, $\alpha > \gamma$ or $\beta > \gamma$, then $m_{\alpha,\beta,\gamma} < 0$.*

(6) *If $\gamma < 0$, $\alpha,\beta > 2\gamma - 1$ or $0 < \gamma < 1$, $-\infty < \alpha,\beta < \infty$, then $m_{\alpha,\beta,\gamma} = -\infty$.*

Theorem 2.3.

(1) *If $\gamma > 1$, $-\infty < \alpha,\beta < \infty$ or $0 < \gamma \leq 1$, $\alpha \leq 2\gamma - 1$, $-\infty < \beta < \infty$ ($\beta \leq 2\gamma - 1$, $-\infty < \alpha < \infty$), then $M_{\alpha,\beta,\gamma} = \pi^2$.*

(2) *If $\gamma < 0$ or $0 < \gamma < 1$, $\alpha,\beta > 2\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*

(3) *If $\gamma < -1$, $\alpha,\beta > 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $M_{\alpha,\beta,\gamma} = R[Q_*,u]$. Moreover, u satisfies equation (2.1) and the integral condition (2.2).*

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