Asymptotic Behaviour of Solutions of Third-Order Differential Equations with Rapid Varying Nonlinearities

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Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a twice continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi(y) = \begin{cases} 0, \\ \text{or } \ +\infty, \end{cases} \qquad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 is equal either to zero or to $\pm\infty$, Δ_{Y_0} is a one-sided neighborhood of the point Y_0.

It follows directly from conditions (2) that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0, \ \ y \in \Delta_{Y_0} \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

By virtue of these conditions, the function φ and its first-order derivative are (see the monograph by M. Maric [10, Chapter 3, § 3.4, Lemmas 3.2, 3.3, pp. 91-92]) rapidly varying as $y \to Y_0$.

For second-order differential equations with the right-hand side the same as in (1), the asymptotic behavior of solutions was studied in [2, 3, 5–7, 10].

In the work of V. M. Evtukhov, N. V. Sharay (see [9]) for the differential equation (1) the questions on the existence and asymptotics of so-called $P_\omega(Y_0, \lambda_0)$ – solutions for $\lambda_0 \in \mathbb{R} \setminus \{0; 1; \frac{1}{2}\}$ were solved.

Definition. A solution y of the differential equation (1) is called $P_\omega(Y_0, \lambda_0)$ – solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} 0, \\ \text{or } \ \pm\infty \end{cases} \quad (k = 1, 2), \qquad \lim_{t \uparrow \omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The aim of the present report is to obtain the asymptotics of $P_\omega(Y_0, \lambda_0)$ – solutions of the differential equation (1) in the special case when $\lambda_0 = 1$. For each such solution, due to a priori asymptotic properties of $P_\omega(Y_0, 1)$ – solutions (see [4, Chapter 3, § 10]), the following relations

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \frac{y'''(t)}{y''(t)} \ \text{ as } \ t \uparrow \omega, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty, \tag{3}$$

hold, where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega = -\infty. \end{cases}$$

Hence, in particular, it follows that $P_\omega(Y_0,1)$ – solution of equation (1) and its derivatives up to the second order inclusive are rapidly varying functions as $t\uparrow\omega$.

Moreover, here and in the sequel, without loss of generality, we assume that

$$\Delta_{Y_0}=\Delta_{Y_0}(y_0),\ \ \text{where}\ \ \Delta_{Y_0}(y_0)=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases} \tag{4}$$

where $y_0\in\Delta_{Y_0}$ is such that $|y_0|<1$ as $Y_0=0$ and $y_0>1$ $(y_0<-1)$ as $Y_0=+\infty$ (as $Y_0=-\infty$).

Let us introduce the necessary auxiliary notations and assume that the definition area of the function φ in equation (1) is determined by formula (4). Further, we put

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$J_0(t)=\int\limits_{A_0}^{t}p_0^{\frac13}(\tau)\,d\tau,\quad \Phi(y)=\int\limits_{B}^{y}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}\,,$$

where $p_0:[a,\omega[\,\to\,]0,+\infty[$ is a continuous or continuously differentiable function such that $p(t)\sim p_0(t)$ as $t\uparrow\omega$,

$$A_0=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^{\omega}p_0^{\frac13}(\tau)\,d\tau<+\infty,\\ a & \text{if } \displaystyle\int\limits_a^{\omega}p_0^{\frac13}(\tau)\,d\tau=+\infty,\end{cases}\qquad B=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=const,\\ y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{s^{\frac23}\varphi^{\frac13}(s)}=\pm\infty.\end{cases}$$

It is clear that the conditions

$$\nu_0\nu_1<0\ \text{ if }\ Y_0=0,\quad \nu_0\nu_1>0\ \text{ if }\ Y_0=\pm\infty,$$

are necessary for the existence of $P_\omega(Y_0,1)$–solutions. Moreover, by virtue of (1), Definition and (3), it is also necessary that the inequalities

$$\alpha_0\nu_1>0,\quad \nu_0\operatorname{sign}y''(t)>0$$

hold.

The entered function Φ keeps a sign on Δ_{Y_0}, tends either to zero or to $\pm\infty$ as $y\to Y_0$ and is increasing on Δ_{Y_0}, since on this interval $\Phi'(y)=y^{-\frac23}\varphi^{-\frac13}(y)>0$. Therefore, there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\longrightarrow\Delta_{Y_0}$, where, by virtue of the second of conditions (2) and the monotonic increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}0,\\ \text{or}\ \ +\infty,\end{cases}$$

$$\Delta_{Z_0}=\begin{cases}[z_0,Z_0[& \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0] & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\Phi(y_0).$$

We also introduce auxiliary functions:

$$q(t)=\frac{(\Phi^{-1}(\alpha_0J_0(t)))'}{\alpha_0J_2(t)},\quad H(t)=\frac{\Phi^{-1}(\alpha_0J_0(t))\varphi'(\Phi^{-1}(\alpha_0J_0(t)))}{\varphi(\Phi^{-1}(\alpha_0J_0(t)))},$$

$$J_1(t)=\int\limits_{A_1}^{t}p_0(\tau)\varphi(\Phi^{-1}(\alpha_0J_0(\tau)))\,d\tau,\quad J_2(t)=\int\limits_{A_2}^{t}J_1(\tau)\,d\tau,$$

where

$$A_1=\begin{cases} t_1 & \text{if } \displaystyle\int\limits_{t_1}^{\omega}p_0(\tau)\varphi\big(\Phi^{-1}(\alpha_0J_0(\tau))\big)\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_{t_1}^{\omega}p_0(\tau)\varphi\big(\Phi^{-1}(\alpha_0J_0(\tau))\big)\,d\tau<+\infty,\end{cases}\qquad t_1\in[a,\omega],$$

$$A_2=\begin{cases} t_1 & \text{if } \displaystyle\int\limits_{t_1}^{\omega}J_1(\tau)d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_{t_1}^{\omega}J_1(\tau)d\tau<+\infty.\end{cases}$$

Note that with the implementation of the properties with regular varying and rapid varying functions [1,11], as well as the results of work [4,8] for equation (1) conditions for the existence of solutions are established.

Theorem 1. *For the existence of $P_\omega(Y_0,1)$ – solutions of the differential equation* (1)*, it is necessary that the inequalities*

$$\alpha_0\nu_1>0,\quad \alpha_0\mu_0J_0(t)<0\ \ as\ \ t\in\,]a,\omega[\,, \tag{5}$$

$$\nu_0\alpha_0<0\ \ if\ \ Y_0=0,\quad \nu_0\alpha_0>0\ \ if\ \ Y_0=\pm\infty \tag{6}$$

and the conditions

$$\frac{\alpha_0J_2(t)}{\Phi^{-1}(\alpha_0J_0(t))}\sim\frac{J_1(t)}{J_2(t)}\sim\frac{J_1'(t)}{J_1(t)}\sim\frac{(\Phi^{-1}(\alpha_0J_0(t)))'}{\Phi^{-1}(\alpha_0J_0(t))}\ \ as\ \ t\uparrow\omega, \tag{7}$$

$$\alpha_0\lim_{t\uparrow\omega}J_0(t)=Z_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)(\Phi^{-1}(\alpha_0J_0(t)))'}{\Phi^{-1}(\alpha_0J_0(t))}=\pm\infty,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_0'(t)}{J_0(t)}=\pm\infty \tag{8}$$

hold. Moreover, each such solution of that kind admits the asymptotic, as $t\uparrow\omega$, representations

$$y(t)=\Phi^{-1}(\alpha_0J_0(t))\Big[1+\frac{o(1)n}{H(t)}\Big], \tag{9}$$

$$y'(t)=\alpha_0(t)J_2(t)[1+o(1)],\quad y''(t)=\alpha_0J_1(t)[1+o(1)]. \tag{10}$$

Theorem 2. *Let $p_0:[a,\omega[\,\to\,]0,+\infty[$ be a continuously differentiable function and along with* (5)–(8) *the conditions*

$$\lim_{t\uparrow\omega}\frac{q'(t)H^{\frac13}(t)J_2(t)}{J_2'(t)}=0,\quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{\frac23}=0$$

hold. Then the differential equation (1) *in case* $\alpha_0\mu_0 > 0$ *has a two-parameter and in case* $\alpha_0\mu_0 < 0$ *has a one-parameter family of* $P_\omega(Y_0,1)$ *– solutions that admit asymptotic, as* $t \uparrow \omega$*, representations* (9) *and moreover, their derivatives of the first and second order satisfy the asymptotic, as* $t \uparrow \omega$*, relations*

$$y'(t) = \alpha_0 J_2(t)\big[q(t) + o((H(t))^{-\frac{2}{3}})\big], \quad y''(t) = \alpha_0 J_1(t)\big[q(t) + o((H(t))^{-\frac{1}{3}})\big].$$

It is possible to notice that in the asymptotic relations (7)

$$\frac{(\Phi^{-1}(\alpha_0 J_0(t)))'}{\Phi^{-1}(\alpha_0 J_0(t))} = \alpha_0\Big(\frac{p_0(t)\varphi(\Phi^{-1}(\alpha_0 J_0(t)))}{\Phi^{-1}(\alpha_0 J_0(t))}\Big)^{\frac{1}{3}}.$$

Therefore, it follows from (7) that

$$J_2(t) = \Big(p_0(t)(\Phi^{-1}(\alpha_0 J_0(t)))^2\varphi(\Phi^{-1}(\alpha_0 J_0(t)))\Big)^{\frac{1}{3}}[1 + o(1)] \ \text{ as } \ t \uparrow \omega,$$

$$J_1(t) = \alpha_0(\Phi^{-1}(\alpha_0 J_0(t)))^{\frac{1}{3}}\Big(p_0(t)\varphi(\Phi^{-1}(\alpha_0 J_0(t)))\Big)^{\frac{2}{3}}[1 + o(1)] \ \text{ as } \ t \uparrow \omega.$$

These relations allow to rewrite the asymptotic relations (10) without integrals.

Theorem 3. *Let* $p_0 : [a,\omega[\,\to\,]0,+\infty[$ *be a continuous function and, along with* (5)–(8)*, the conditions*

$$\lim_{t\uparrow\omega}[1 - q(t)]H^{\frac{2}{3}}(t) = 0, \qquad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^{\frac{2}{3}} = 0$$

hold. Then the differential equation (1) *in case* $\alpha_0\mu_0 > 0$ *has a two-parameter family, and in case* $\alpha_0\mu_0 < 0$ *has a one-parameter family of* $P_\omega(Y_0,1)$*– solutions, admitting as* $t \uparrow \omega$ *the asymptotic representations* (9) *and*

$$y'(t) = \alpha_0 J_2(t)\Big[1 + \frac{o(1)}{H^{\frac{2}{3}}(t)}\Big], \quad y''(t) = \alpha_0 J_1(t)\Big[1 + \frac{o(1)}{H^{\frac{1}{3}}(t)}\Big].$$

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