A Condition for the Solvability of the Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems the Lower Triangular Representation of Mean Value of Coefficient Matrix

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In 1955 J. Kurzweil and O. Vejvoda proved that the system of almost periodic differential equations can have an almost periodic solution such that the intersection of the frequency modules of the solution and the right-hand side is trivial [3]. In what follows, such almost periodic solutions will be called strongly irregular, the frequency spectrum – asynchronous, and the described vibrations – asynchronous [1, 4]. Various aspects of control theory for ordinary differential systems of almost periodic equations were studied in a number of works (see, for example, [5] and others), the essential a feature of which is the consideration the regular case, when the frequency of the system itself and its solution coincide.

Now we will study the solvability of the control problem of the asynchronous spectrum of linear almost periodic systems for which the mean value of the coefficient matrix is lower triangular. Let's consider a linear non-stationary control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where x is the phase vector, u is the input, B is the constant $n \times n$-matrix under control, $A(t)$ is a continuous almost periodic matrix with a modulus of frequencies $\operatorname{Mod}(A)$. Suppose that the control is specified in the form of a linear feedback in the phase variables

$$u = U(t)x \tag{2}$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ (feedback coefficient), the frequency modulus of which is contained in the frequency modulus of the coefficient matrix, i.e.

$$\operatorname{Mod}(U) \subseteq \operatorname{Mod}(A).$$

It is required to obtain conditions on the right-hand side of system (1) such that for any choice of the feedback coefficient from the indicated admissible set, the closed-loop system

$$\dot{x} = (A(t) + BU(t))x, \tag{3}$$

has a strongly irregular almost periodic solution, the frequency spectrum of which contains a given subset (target set).

Let L be the target frequency set. We will assume that

$$\operatorname{rank} B = r < n \ \ (n - r = d). \tag{4}$$

In this case, there is a constant non-singular real $(n\times n)$-matrix S such that in the matrices $D=SB$ the first d columns are zero, while the rest r columns are linearly independent.

Let us introduce the transformation of phase variables

$$y=Sx, \tag{5}$$

which transform system (3) to the system

$$\dot y=(C(t)+DV(t))y, \tag{6}$$

where

$$C(t)=SA(t)S^{-1},\quad V(t)=U(t)S^{-1},\quad D=QB.$$

System (6) has a strongly irregular almost periodic solution if and only if this solution satisfies the system

$$\dot y=(\widehat{C}+D\widehat{V})y,\quad (\widetilde{C}(t)+D\widetilde{V}(t))y=0, \tag{7}$$

where sign "$\widehat{\ }$" denotes a averaging, for example,

$$\widehat{C}=\lim_{T\to+\infty}\frac{1}{T}\int\limits_0^T C(s)\,ds,\quad \widetilde{C}(t)=C(t)-\widehat{C},\quad \widetilde{V}(t)=V(t)-\widehat{V}.$$

Let us denote the matrix composed of the last r rows of the matrix D, by $D_{r,n}$. It follows from the construction of the matrix D that the following condition is fulfilled

$$\operatorname{rank}D_{r,n}=r. \tag{8}$$

The rank condition (8) means that the matrix $D_{r,n}$ rows are linearly independent. Since their number is less than the number of columns, then adding any columns to such a matrix does not change its rank.

Let us represent the matrix of coefficients $C(t)$ in the block form, corresponding to the structure of the matrix D. Let $C^{(11)}_{d,d}(t)$, $C^{(21)}_{r,d}(t)$ – its upper and lower left, and $C^{(12)}_{d,r}(t)$, $C^{(22)}_{r,r}(t)$ – its upper and lower right blocks (the lower indices indicate the dimensionality). According to this representation, the averaged matrix $\widehat{C}$ will be decomposed into four blocks of the same dimensions $\widehat{C}^{(11)}_{d,d}$, $\widehat{C}^{(21)}_{r,d}$, $\widehat{C}^{(12)}_{d,r}$, $\widehat{C}^{(22)}_{r,r}$.

Taking into account the structure of the matrix D and the block representation averaging of the matrix of coefficients $C(t)$, we write system (7) in the form

$$\begin{gathered}\dot y^{[d]}=\widehat{C}^{(11)}_{d,d}y^{[d]}+\widehat{C}^{(12)}_{d,r}y_{[r]},\quad \dot y_{[r]}=\big(\widehat{C}^{(21)}_{r,d}+D_{r,n}\widehat{V}_{n,d}\big)y^{[d]}+\big(\widehat{C}^{(22)}_{r,r}+D_{r,n}\widehat{V}_{n,r}\big)y_{[r]},\\ \widetilde{C}^{(11)}_{d,d}(t)y^{[d]}+\widetilde{C}^{(12)}_{d,r}(t)y_{[r]}=0,\quad \big(\widetilde{C}^{(21)}_{r,d}(t)+D_{r,n}\widetilde{V}_{n,d}(t)\big)y^{[d]}+\big(\widetilde{C}^{(22)}_{r,r}(t)+D_{r,n}\widetilde{V}_{n,r}(t)\big)y_{[r]}=0,\end{gathered} \tag{9}$$

where

$$\begin{gathered}y=\operatorname{col}(y^{[d]},y_{[r]}),\quad y^{[d]}=\operatorname{col}(y_1,\dots,y_d),\quad y_{[r]}=\operatorname{col}(y_{d+1},\dots,y_n),\\ \widehat{V}=\{\widehat{V}_{n,d}\widehat{V}_{n,r}\},\quad \widetilde{V}(t)=\{\widetilde{V}_{n,d}(t)\widetilde{V}_{n,r}(t)\}\end{gathered}$$

are the corresponding‘ representation of the stationary and oscillatory components of the matrix $V(t)$.

Thus, it is true

Lemma. *If conditions* (4), (8) *are fulfilled, systems* (3) *and* (9) *are equivalent in the sense of existence of strongly irregular almost periodic solutions.*

Suppose that the averaging of the matrix of coefficients of the original system with using the transforming matrix S is reduced to the lower-triangular form. In other words, this means that the matrix $\widehat{C}$ has the form

$$\widehat{C}=\begin{pmatrix} \widehat{c}_{11} & 0 & 0\dots & 0\\ \widehat{c}_{21} & \widehat{c}_{22} & 0\dots & 0\\ \dots & \dots & \dots & \dots\\ \widehat{c}_{n1} & \widehat{c}_{n2} & \widehat{c}_{n3}\dots & \widehat{c}_{nn}\end{pmatrix}. \tag{10}$$

Let's give the conditions to solve the posed problem. Taking into account the lemma, this problem is reduced to finding conditions for the existence of strongly irregular almost periodic solutions $y=y(t)=\operatorname{col}(y^{[d]}(t),y_{[r]}(t))$ with the frequencies L of system (9).

We have

Theorem. *The control problem of asynchronous spectrum of system* (1), (4), (10) *with the target set L is solvable if and only if the conditions*

$$\operatorname{rank}_{\rm col} C_{12}=r_1<r$$

and

$$|L|\le\Big[\frac{r-r_1}{2}\Big]$$

are satisfied.

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References

[1] A. Demenchuk, *Asynchronous Oscillations in Differential Systems. Conditions of Existence and Control.* (Russian) Lambert Academic Publishing, Saarbrucken, 2012.

[2] F. R. Gantmaher, *The theory of matrices.* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953.

[3] J. Kurzweil and O. Vejvoda, On the periodic and almost periodic solutions of a system of ordinary differential equations. (Russian) *Czechoslovak Math. J.* **5(80)** (1955), 362–370.

[4] D. I. Penner, D. B. Duboshinskiĭ, M. I. Kozakov, A. S. Vermel' and Yu. V. Galkin, Asynchronous excitation of undamped oscillations. (Russian) *Uspehi Fiz. Nauk.* **109** (1973), 402-406; translation in *Sov. Phys. Usp.* **16** (1973), no. 1, 158–160.

[5] N. K. Popova and V. N. Tarasov, Convergence of a modification of Kelly's method. (Russian) *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **1985**, vyp. 1, 117–119.