# About the Reduction of the Linear Noetherian Difference Algebraic Boundary Value Problem to the Noncritical Case

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We investigate the problem of finding bounded solutions [2,3,6]

$$z(k)\in\mathbb{R}^n,\ \ k\in\Omega:=\{0,,2,\dots,\omega\}$$

of the linear Noetherian ($n\neq\upsilon$) boundary value problem for a system of linear difference-algebraic equations [2,6]

$$A(k)z(k+1)=B(k)z(k)+f(k),\ \ \ell z(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^{\upsilon}; \tag{1}$$

here $A(k),B(k)\in\mathbb{R}^{m\times n}$ are bounded matrices and $f(k)$ are real bounded column vectors,

$$\ell z(\,\cdot\,):\mathbb{R}^n\to\mathbb{R}^{\upsilon}$$

is a linear bounded vector functional defined on a space of bounded functions. We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m=n$. It can be square, but singular. The problem of finding bounded solutions $z(k)$ of a boundary value problem for a linear non-degenerate ($\det B(k)\neq 0$, $k\in\Omega$) system of first-order difference equations

$$z(k+1)=B(k)z(k)+f(k),\ \ \ell z(\,\cdot\,)=\alpha\in\mathbb{R}^{\upsilon}$$

was solved by A. A. Boichuk [2]. We investigate the problem of finding bounded solutions of the linear Noetherian boundary value problem for a system of linear difference-algebraic equations (1) in case

$$1\leq\operatorname{rank}A(k)=\sigma_0,\ \ k\in\Omega.$$

As is known [1,14], any $(m\times n)$-matrix $A(k)$ can be represented in a definite basis in the form

$$A(k)=R_0(k)\cdot J_{\sigma_0}\cdot S_0(k),\quad J_{\sigma_0}:=\begin{pmatrix} I_{\sigma_0} & O\\ O & O\end{pmatrix};$$

here, $R_0(k)$ and $S_0(k)$ are nonsingular matrices. The nonsingular change of the variable

$$y(k+1)=S_0(k)z(k+1)$$

reduces system (1) to the form [12]

$$A_1(k)\varphi(k+1)=B_1(k)\varphi(k)+f_1(k); \tag{2}$$

Under the condition [14], when

$$P_{A^*}(k)\neq 0,\ \ P_{A_1^*}(k)\equiv 0, \tag{3}$$

we arrive at the problem of construction of solutions of the linear difference-algebraic system

$$\varphi(k+1)=A_1^+(k)B_1(k)\varphi(k)+\mathfrak{F}_1(k,\nu_1(k)),\ \ \nu_1(k)\in\mathbb{R}^{\rho_1}; \tag{4}$$

here,

$$\mathfrak{F}_1(k,\nu_1(k)) := A_1^+(k) f_1(k) + P_{A_{\varrho_1}}(k)\nu_1(k),$$

$\nu_1(k) \in \mathbb{R}^{\rho_1}$ is an arbitrary bounded vector function, $A_1^+(k)$ is a pseudoinverse (by Moore–Penrose) matrix [3]. In addition, $P_{A_1^*(k)}$ is a matrix-orthoprojector [3]: $P_{A_1^*}(k) : \mathbb{R}^{\sigma_0} \to \mathbb{N}(A_1^*(k))$, $P_{A_{\rho_1}}(k)$ is an $(\rho_0 \times \rho_1)$-matrix composed of $\rho_1$ linearly independent columns of the $(\rho_0 \times \rho_0)$-matrix-orthoprojector: $P_{A_1}(k) : \mathbb{R}^{\rho_0} \to \mathbb{N}(A_1(k))$. By analogy with the classification of pulse boundary-value problems [3, 7, 8] we say in the (3), that, for the linear difference-algebraic system (1), the first-order degeneration holds. Thus, the following lemma is proved [12].

**Lemma.** *The first-order degeneration difference-algebraic system* (1) *has a solution of the form*

$$z(k,c_{\rho_0}) = X_1(k)\, c_{\rho_0} + K[f(j),\nu_1(j)](k), \ \ c_{\rho_0} \in \mathbb{R}^{\rho_0};$$

*which depends on the arbitrary continuous vector-function $\nu_1(k) \in \mathbb{R}^{\rho_1}$, where $X_1(k)$ is fundamental matrix, $K[f(j),\nu_1(j)](k)$ is the generalized Green operator of the Cauchy problem for the linear difference-algebraic system* (1)*.*

Denote the vector $\nu_1(k) := \Psi_1(k)\gamma$, $\gamma \in \mathbb{R}^{\theta}$; here, $\Psi_1(k) \in \mathbb{R}^{\rho_1 \times \theta}$ is an arbitrary bounded full rank matrix. Generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1) of the form

$$K[f(j),\nu_1(j)](k) = K[f(j)](k) + K[\Psi_1(j)](k)\,\gamma;$$

here,

$$K[\Psi_1(j)](k) := S_0^{-1}(k-1) P_{D_{\rho_0}} \mathcal{K}[\Psi_1(s))](k),$$

and

$$\begin{aligned}\mathcal{K}[\Psi_1(j)](0) := 0, \ \mathcal{K}[\Psi_1(j)](1) := P_{A_{\rho_1}}(0)\Psi_1(0),\\ \mathcal{K}[\Psi_1(j)](2) := A_1^+(1)B_1(1)\mathcal{K}[\Psi_1(j)](1) + P_{A_{\rho_1}}(1)\Psi_1(1), \ \dots,\\ \mathcal{K}[\Psi_1(j)](k+1) := A_1^+(k)B_1(k)\mathcal{K}[\Psi_1(j)](k) + P_{A_{\rho_1}}(k)\Psi_1(k).\end{aligned}$$

Denote the matrix

$$\mathcal{D}_1 := \big\{Q_1;\ \ell K[\Psi_1(j)](\,\cdot\,)\big\} \in \mathbb{R}^{\upsilon \times (\rho_0+\theta)}.$$

Substituting the general solution of the system of the linear difference-algebraic equations (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$\mathcal{D}_1\, \check{c} = \alpha - \ell K[A^+(j) f(j)](\,\cdot\,), \ \ \check{c} := \operatorname{col}(c_{\rho_0},\gamma) \in \mathbb{R}^{\rho_0+\theta}. \tag{5}$$

Equation (5) is solvable iff

$$P_{\mathcal{D}_1^*}\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\} = 0. \tag{6}$$

Here, $P_{\mathcal{D}_1^*}$ is a matrix-orthoprojector: $P_{\mathcal{D}_1^*} : \mathbb{R}^{\upsilon} \to \mathbb{N}(\mathcal{D}_1^*)$. In this case, the general solution of equation (5)

$$\check{c} = \mathcal{D}_1{}^+\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\} + P_{\mathcal{D}_1}\delta, \ \ \delta \in \mathbb{R}^{\rho_0+\theta}$$

determines the general solution of the boundary-value problem (1)

$$\begin{aligned}z(k,\delta) = \big\{X_1(k); K[\Psi_1(j)](k)\big\}\mathcal{D}_1^+\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\}\\ + K[f(j)](k) + \big\{X_1(k); K[\Psi_1(j)](k)\big\}P_{\mathcal{D}_1}\delta.\end{aligned}$$

Here, $P_{\mathcal{D}_1}$ is a matrix-orthoprojector: $P_{\mathcal{D}_1} : \mathbb{R}^{\rho_0+\theta} \to \mathbb{N}(\mathcal{D}_1)$. Thus, the following theorem is proved [12].

**Corollary.** *The problem of finding bounded solutions of a system of linear difference-algebraic equations* (1) *in the case of first-order degeneracy, under condition* (3)*, in the case of first-order degeneracy for a fixed full rank bounded matrix* $\Psi_1(k)$*, has a solution of the form*

$$z(k, c_{\rho_0}) = X_1(k)\, c_{\rho_0} + K\big[f(j), \nu_1(j)\big](k), \ \ c_{\rho_0} \in \mathbb{R}^{\rho_0}.$$

*Under condition* $P_{Q_1^*} \neq 0$, $P_{\mathcal{D}_1^*} = 0$*, the general solution of the difference-algebraic boundary value problem* (1)

$$z(k, c_r) = X_r(k)c_r + G\big[f(j); \Psi_1(j); \alpha\big](k), \ \ c_r \in \mathbb{R}^r$$

*is determined by the Green operator of a difference-algebraic boundary value problem* (1)

$$G\big[f(j); \Psi_1(j); \alpha\big](k) := K[f(j)](k) + \big\{X_1(k); K[\Psi_1(j)](k)\big\}\mathcal{D}_1^+\big\{\alpha - \ell K[f(j)](\,\cdot\,)\big\}.$$

*The matrix* $X_r(k)$ *is composed of* $r$ *linearly independent columns of the matrix*

$$\big\{X_1(k); K[\Psi_1(j)](k)\big\}P_{\mathcal{D}_1}.$$

Under condition $P_{\mathcal{D}_1^*} \neq 0$, we say that the difference-algebraic boundary-value problem (1) in the case of first-order degeneracy is a critical case, and vice versa: under condition $P_{Q_1^*} \neq 0$, $P_{\mathcal{D}_1^*} = 0$, we say that the difference-algebraic boundary-value problem (1) is reduced to the non-critical case.

**Example.** The requirements of the proved Corollary 1 satisfy the problem of construction solutions of the difference-algebraic boundary-value problem

$$A\, z(k+1) = B(k)\, z(k) + f(k), \quad \ell z(\,\cdot\,) := M\,(z(0) - z(3)) = 0, \ \ k = 0, 1, 2, 3, \tag{7}$$

here

$$A := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k+1 & 0 & 0 \end{pmatrix}, \quad f(k) := \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}, \quad M := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first-order degeneration difference-algebraic system (7) has a solution of the form

$$z(k) = X_r(k)\, c_r + G\big[f(j), \nu_1(j), \alpha\big](k), \quad G\big[f(j), \nu_1(j), \alpha\big](k) = K\big[f(j), \nu_1(j)\big](k), \ \ c_r \in \mathbb{R}^1,$$

where

$$X_r(k) := X_1(k)P_{Q_r} = P_{Q_r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nu_1(k) := 0, \ \ k \in \Omega := \{0, 1, 2, 3\},$$

in addition,

$$K\big[f(j), \nu_1(j)\big](0) = -\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad K\big[f(j), \nu_1(j)\big](1) = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \end{pmatrix},$$

$$K\big[f(j), \nu_1(j)\big](2) = \frac{1}{3}\begin{pmatrix} 0 \\ -1 \\ 6 \\ 3 \end{pmatrix}, \quad K\big[f(j), \nu_1(j)\big](0) = \frac{1}{4}\begin{pmatrix} 0 \\ -1 \\ 12 \\ 8 \end{pmatrix}.$$

The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [2–4, 10] onto nonlinear difference-algebraic boundary-value problems. On the other hand, in the case of nonsolvability, the difference-algebraic boundary-value problems can be regularized analogously [9, 15]. The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [5, 11, 12] onto nonlinear difference-algebraic boundary-value problems with variable rank of leading coefficient matrix an analogously to [13] an inverse problem to the Cauchy problem for the difference-algebraic equation.

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