Linear Differential-Algebraic Boundary Value Problem with Pulse Influence

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The study of differential-algebraic boundary value problems was initiated in the works of K. Weierstrass, N. N. Luzin and F. R. Gantmacher. Systematic study of differential-algebraic boundary value problems is devoted to the works of S. Campbell, Yu. E. Boyarintsev, V. F. Chistyakov, A. M. Samoilenko, M. O. Perestyuk, V. P. Yakovets, O. A. Boichuk, A. Ilchmann and T. Reis [3]. The study of the differential-algebraic boundary value problems is associated with numerous applications of such problems in the theory of nonlinear oscillations, in mechanics, biology, radio engineering, theory of control, theory of motion stability. At the same time, the study of differential algebraic boundary value problems is closely related to the study of pulse boundary value problems for differential equations, initiated by M. O. Bogolybov, A. D. Myshkis, A. M. Samoilenko, M. O. Perestyk and O. A. Boichuk. Consequently, the actual problem is the transfer of the results obtained in the articles by S. Campbell, A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk on pulse linear boundary value problems for differential-algebraic equations, in particular finding the necessary and sufficient conditions for the existence of the desired solutions, and also the construction of the Green operator of the Cauchy problem and the generalized Green operator of a pulse linear boundary value problem for a differential-algebraic equation [2, 3, 5].

In this article we found the conditions of the existence and constructive scheme for finding the solutions

$$z(t) \in \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\}$$

of the linear Noetherian differential-algebraic boundary value problem for a differential-algebraic equation with the singular impulse action [2–5, 11]

$$A(t)z'(t) = B(t)z(t) + f(t), \ \ t \neq \tau_i, \tag{1}$$

$$\Delta z(\tau_i) = S_i z(\tau_i - 0) + a_i, \ \ i = 1, 2, \dots, p, \ \ \ell z(\,\cdot\,) = \alpha \in \mathbb{R}^q, \tag{2}$$

where $A(t), B(t) \in \mathbb{C}_{m\times n}[a,b]$, $f(t) \in \mathbb{C}[a,b]$, $\ell z(\,\cdot\,)$ is a linear bounded functional:

$$\ell z(\,\cdot\,) : \mathbb{C}^1\{[a,b] \setminus \{\tau_i\}_I\} \to \mathbb{R}^q.$$

We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m = n$. It can be square, but singular. The proposed scheme of the research of the linear differential-algebraic boundary value problem for a differential-algebraic equation with impulse action in the critical case in this article can be transferred to the linear differential-algebraic boundary value problem for a differential-algebraic equation with a singular impulse action. The above scheme of the analysis of the seminonlinear differential-algebraic boundary value problems with an impulse action generalizes the results of S. Campbell, A. M. Samoilenko, M. O. Perestyuk and O. A. Boichuk and can be used for proving the solvability and constructing solutions of weakly nonlinear boundary value problems with a singular impulse action in the critical and noncritical cases [2–6, 8, 11]. For the case in which all

$$A(t)\equiv I_n,\ \ I_n+S_i,\ \ i=1,2,\dots,p$$

are non-degenerate matrices, we obtain the problem investigated in [2, 3]; in particular, if

$$A(t)\equiv I_n,\quad S_i=0,\ \ i=1,2,\dots,p,$$

then we have the problem considered in [12]. If

$$S_i=0,\ \ a_i=0,\ \ \ell z(\,\cdot\,):=M_i(\tau_i-0)+N_iz(\tau_i+0),\ \ i=1,2,\dots,p,$$

then we obtain the problem analyzed in [10]. If $A(t)\equiv I_n$ and I_n+S_i are degenerate matrices for some i, then we have a degenerate impulse action [5, 6].

The analysis of differential-algebraic equations with the help of the central canonical form, perfect pairs, and matrix triplets was made in the monographs [3, 4]. Sufficient conditions for the reducibility of a differential-algebraic linear system to the central canonical form were obtained by A. M. Samoilenko and V. P. Yakovets. In papers [8, 11], sufficient solvability conditions, as well as the procedure of constructing the generalized Green operator for the linear differential-algebraic boundary-value problem (1) without making use of the central canonical form, perfect pairs, and matrix triplets, were proposed.

Provided that the differential-algebraic system (1) with the constant-rank matrix $A(t)$ satisfies the condition

$$P_{A^*(t)}\equiv 0. \tag{3}$$

Then, in the case (3), the differential-algebraic system (1) has a solution, which can be written in the form [8, 11]

$$z(t,c)=X_0(t)c+K_0\big[f(s),\nu_0(s)\big](t),\ \ c\in\mathbb{R}^n,$$

where

$$K_0\big[f(s),\nu_0(s)\big](t):=X_0(t)\int\limits_a^t X_0^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds,\ \ a\le t<\tau_1$$

is the generalized Green operator of the Cauchy problem $z(a)=0$ for the differential-algebraic system (1), $X_0(t)$ is normal fundamental matrix:

$$X_0'(t)=A^+(t)B(t)X_0(t),\ \ X_0(a)=I_n$$

and

$$\mathfrak{F}_0(t,\nu_0(t)):=A^+(t)f(t)+P_{A_{\rho_0}}(t)\nu_0(t),$$

$A^+(t)$ is a pseudoinverse matrix, $P_{A^*(t)}$ is matrix-orthoprojectors [3]:

$$P_{A^*}(t):\mathbb{R}^m\to\mathbb{N}(A^*(t)),$$

$P_{A_{\rho_0}}(t)$ is an $(n\times\rho_0)$-matrix composed of ρ_0 linearly independent columns of the $(n\times n)$-matrix-orthoprojector [3]:

$$P_A(t):\mathbb{R}^n\to\mathbb{N}(A(t)).$$

The solvability of the differential-algebraic boundary-value problem (1), (2) substantially depends on the choice of the continuous vector function $\nu_0(t)$. In the case (3) the differential-algebraic system (1) has a solution which can be written in the form [5, 6, 9]

$$z(t,c) = X_1(t)c + K_1\big[f(s),\nu_0(s)\big](t), \ \ c \in \mathbb{R}^n,$$

where

$$X_1(t) = X_0(t)X_0^{-1}(\tau_1)(I_n + S_1)X_0(\tau_1), \ \ \tau_1 \le t < \tau_2,$$

and

$$K_1\big[f(s),\nu_0(s)\big](t) := X_0(t)X_0^{-1}(\tau_1)(I_n+S_1)K_0\big[f(s),\nu_0(s)\big](\tau_1) \\ + X_0(t)X_0^{-1}(\tau_1)\,a_1 + X_0(t)\int\limits_{\tau_1}^{t} X_0^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds, \ \ \tau_1 \le t < \tau_2.$$

Denote the matrix

$$X_p(t) = X_0(t)X_0^{-1}(\tau_p)(I_n + S_p)X_{p-1}(\tau_p), \ \ \tau_p \le t \le b,$$

and

$$K_p\big[f(s),\nu_0(s)\big](t) := X_0(t)X_0^{-1}(\tau_p)(I_n+S_p)K_{p-1}\big[f(s),\nu_0(s)\big](\tau_p) \\ + X_0(t)X_0^{-1}(\tau_p)\,a_p + X_0(t)\int\limits_{\tau_p}^{t} X_0^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds, \ \ \tau_p \le t \le b$$

is the generalized Green operator of the Cauchy problem for the differential-algebraic system (1) with the singular impulse action (2). Thus, the following lemma is proved.

Lemma 1. *In the case* (3) *the differential-algebraic system* (1) *with the singular impulse action* (2) *with the constant-rank matrix* $A(t)$ *has a solution which can be written in the form*

$$z(t,c) = X(t)\,c + K\big[f(s),\nu_0(s)\big](t), \ \ c \in \mathbb{R}^n,$$

where

$$X(t) = \begin{cases} X_0(t), & a \le t < \tau_1, \\ \dots\dots\dots\dots\dots\dots \\ X_p(t), & \tau_p \le t \le b, \end{cases}$$

and

$$K\big[f(s),\nu_0(s)\big](t) = \begin{cases} K_0\big[f(s),\nu_0(s)\big](t), & a \le t < \tau_1, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ K_p\big[f(s),\nu_0(s)\big](t), & \tau_p \le t \le b \end{cases}$$

is the generalized Green operator of the Cauchy problem for the differential-algebraic system (1) *with the singular impulse action* (2).

If and only if the condition

$$P_{Q_d^*}\{\alpha - \ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\} = 0 \tag{4}$$

is satisfied, the solution of the differential-algebraic boundary-value problem (1), (2) which can be written in the form

$$z(t,c_r)=X_r(t)c_r+X(t)Q^+\{\alpha-\ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\}+K\big[f(s),\nu_0(s)\big](t),\ \ c_r\in\mathbb{R}^r.$$

Here $X_r(t):=X(t)P_{Q_r}$ is a fundamental matrix of the boundary-value problem (1), (2)

$$P_{Q^*}:\mathbb{R}^q\to\mathbb{N}(Q^*),\ \ P_Q:\mathbb{R}^n\to\mathbb{N}(Q),\ \ Q:=\ell X(\,\cdot\,)\in\mathbb{R}^{q\times n}$$

are matrices-orthoprojectors [3], $P_{Q_r}\in\mathbb{R}^{q\times r}$ is an $(n\times r)$-matrix composed of r linearly independent columns of the orthoprojector P_Q. Thus, the following lemma is proved.

Lemma 2. *In the case* (3) *the differential-algebraic system* (1) *with the singular impulse action* (2) *with the constant-rank matrix* $A(t)$ *has a solution which can be written in the form*

$$z(t,c)=X(t)c+K\big[f(s),\nu_0(s)\big](t),\ \ c\in\mathbb{R}^n$$

is the generalized Green operator of the Cauchy problem for the system (1) *with the singular impulse action* (2)*. If and only if condition* (4) *holds, the solution of the differential-algebraic system* (1) *with the singular impulse action* (2)

$$z(t,c_r)=X_r(t)c_r+G\big[f(s);\nu_0(s);\alpha\big](t),\ \ c_r\in\mathbb{R}^r,$$

determines the generalized Green operator of the differential-algebraic system (1) *with the singular impulse action* (2)

$$G\big[f(s);\nu_0(s);\alpha\big](t):=X(t)Q^+\big\{\alpha-\ell K\big[f(s),\nu_0(s)\big](\,\cdot\,)\big\}+K\big[f(s),\nu_0(s)\big](t).$$

The above scheme of the analysis of the boundary value problems with the impulse action (1), (2) generalizes the results of [2–6,9] and can be used for proving the solvability and constructing solutions of weakly nonlinear boundary value problems with singular impulse action in the critical and noncritical cases [1, 3, 7]. The results of the proved Lemmas 1 and 2 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [3, 4].

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