# Boundary Value Problems for Systems of Singular Integral-Differential Equations of Fredholm Type with Degenerate Kernel

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We investigate the problem of finding solutions [1]

$$y(t)\in\mathbb{D}^2[a;b],\ \ y'(t)\in\mathbb{L}^2[a;b]$$

of linear Noetherian ($n\neq\upsilon$) boundary value problem for a system of linear integral-differential equations of Fredholm type with degenerate kernel

$$A(t)y'(t)=B(t)y(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds+f(t),\ \ \ell y(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^p. \tag{1}$$

We seek a solution of the Noetherian boundary value problem (1) in a small neighborhood of the solution

$$y_0(t)\in\mathbb{D}^2[a;b],\ \ y_0'(t)\in\mathbb{L}^2[a;b]$$

of the generating problem

$$A(t)y_0'(t)=B(t)y_0(t)+f(t),\ \ \ell y_0(\,\cdot\,)=\alpha. \tag{2}$$

Here

$$A(t),B(t)\in\mathbb{L}^2_{m\times n}[a;b]:=\mathbb{L}^2[a;b]\otimes\mathbb{R}^{m\times n},\quad \Phi(t)\in\mathbb{L}^2_{m\times q}[a;b],\ \ f(t)\in\mathbb{L}^2[a;b].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m\neq n$. It can be square, but singular. Assume that the function $F(y(t),y'(t),t)$ is linear with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$. In addition, we assume that the function $F(y(t),y'(t),t)$ is continuous in the independent variable $t$ on the segment $[a,b]$;

$$\ell y(\,\cdot\,):\mathbb{D}^2[a;b]\to\mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[a;b]$. The problem of finding solutions of the boundary value problem (1) in case $A(t)=I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [12]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk.

Provided that the differential-algebraic system (2) with the constant-rank matrix $A(t)$ satisfies the conditions of the theorem from the paper [13, p. 15] in the case of $p$-order degeneration. Then, in the case of the $p$-order degeneration, the differential-algebraic system (2) has a solution which can be written the form

$$y_0(t,c_{\rho_{p-1}})=X_p(t)c_{\rho_{p-1}}+K[f(s),\nu_p(s)](t),\ \ c_{\rho_{p-1}}\in\mathbb{R}^{\rho_{p-1}}.$$

The solvability of the differential-algebraic boundary-value problem (2) substantially depends on the choice of the continuous vector function $\nu_p(t)$. If and only if condition

$$P_{Q^*}\big\{\alpha - \ell K[f(s),\nu_p(s)](\,\cdot\,)\big\} = 0 \tag{3}$$

is satisfied, the solution of the differential-algebraic boundary-value problem (2), namely,

$$y_0(t,c_r) = X_r(t)c_r + G[f(s);\alpha](t), \ \ c_r \in \mathbb{R}^r$$

determines the generalized Green operator of the differential-algebraic boundary-value problem (2)

$$G[f(s);\alpha](t) := X_p(t)\,Q^+\big\{\alpha - \ell K[f(s),\nu_p(s)](\,\cdot\,)\big\} + K[f(s),\nu_p(s)](t),$$

where $K[f(s),\nu_p(s)](t)$ is the generalized Green operator of the Cauchy problem $z(a)=0$ for the differential-algebraic system (2). Here $X_r(t) := X_p(t)P_{Q_r}$, $X_p(t)$ is fundamental matrix of the differential-algebraic system (2),

$$P_{Q^*} : \mathbb{R}^{\upsilon} \to \mathbb{N}(Q^*), \ \ P_Q : \mathbb{R}^{\rho_{p-1}} \to \mathbb{N}(Q), \ \ Q := \ell X_p(\,\cdot\,) \in \mathbb{R}^{\upsilon\times\rho_{p-1}}$$

are matrix-orthoprojectors [1, 2, 13], $P_{Q_r} \in \mathbb{R}^{\upsilon\times r}$ is an $(\rho_{p-1}\times r)$-matrix composed of $r$ linearly independent columns of the orthoprojector $P_Q$. Thus, the following lemma is proved [13].

**Lemma 1.** *Provided that the differential-algebraic system* (2) *with the constant-rank matrix* $A(t)$ *satisfies the conditions of the theorem from the paper* [13, p. 15] *in the case of p-order degeneration. Then, in the case of the p-order degeneration, the differential-algebraic system* (2) *has a solution which can be written in the form*

$$y_0(t,c_{\rho_{p-1}}) = X_p(t)c_{\rho_{p-1}} + K[f(s),\nu_p(s)](t), \ \ c_{\rho_{p-1}} \in \mathbb{R}^{\rho_{p-1}}.$$

*In the critical case* $P_{Q^*} \neq 0$ *the singular differential-algebraic boundary value problem* (2) *is solvable iff* (3) *holds. In the critical case the singular differential-algebraic boundary value problem* (2) *has a solution of the form*

$$y_0(t,c_r) = X_r(t)c_r + G[f(s);\alpha](t), \ \ X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

*which depends on the arbitrary vector-function* $\nu_p(t) \in \mathbb{L}^2[a;b]$*. Here,* $P_{Q_r}$ *is an* $(p\times r)$*-matrix composed of* $r$ *linearly independent columns of the* $(p\times p)$*-matrix-orthoprojector:* $P_Q : \mathbb{R}^p \to \mathbb{N}(Q)$*;*

$$G[f(s);\alpha](t) := X(t)Q^+\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} + K[f(s)](t)$$

*is the generalized Green operator of the linear integral-differential problem* (1)*;*

$$K[f(s)](t) := X(t)\int_a^t X^{-1}(s)\,\mathfrak{F}_0(s,\nu_0(s))\,ds$$

*is the generalized Green operator of the Cauchy problem for the integral-differential system* (2)*.*

The results of the proved Lemma 1 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [1, 2].

Denote the matrix

$$\Psi(t) := K[\Phi(s)](t) \in \mathbb{D}^2_{n\times q}[a;b]$$

and $P_{R_\omega} \in \mathbb{R}^{\rho_{p-1}\times\omega}$ matrix composed of $\omega$ linearly independent columns of the matrix-orthoprojector

$$P_R : \mathbb{R}^q \to \mathbb{N}(R), \ \ R := \ell\,\Psi(\,\cdot\,).$$

Thus, the following theorem has been proved [10].

**Theorem 1.** *Provided that the differential-algebraic system* (2) *with the constant-rank matrix* $A(t)$ *satisfies the conditions of the theorem from the paper [13, c. 15]. In the critical case* $(P_{Q^*} \neq 0)$ *under condition* (3) *singular* $(P_{A^*}(t)) \neq 0$ *integral-differential boundary value problem* (2) *has a solution of the form*

$$y_0(t, c_r) = X_r(t)c_r + G[f(s); \alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

*which depends on the arbitrary vector-function* $\nu_p(t) \in \mathbb{L}^2[a; b]$. *The singular integral-differential boundary value problem* (1) *has a solution of the form*

$$y(t) = y_0(t, c_r) + x(t), \quad x(t) = X_p(t)v + \Psi(t)u,$$

$$u(c_r) = P_{Q_r}c_r, \quad v(c_\omega) = P_{R_\omega}c_\omega, \quad \check{c} := \begin{pmatrix} c_r \\ c_r \\ c_1 \end{pmatrix},$$

*where*

$$\varphi(\check{c}) := u(c_0) - \int\limits_a^b F\Big(y_0(t, c_r) + x\big(t, u(c_r), v(c_\omega)\big), y_0'(t, c_r) + x'\big(t, u(c_r), v(c_\omega)\big), t\Big)\, dt = 0. \tag{4}$$

*Suppose that for equation* (4) *the following conditions are satisfied:*

1. *A non-linear vector-function* $\varphi(\check{c})$, *twice continuously differentiable with respect to* $\check{c}$ *in some region* $\Omega \subseteq \mathbb{R}^{2r+\omega}$, *in a neighborhood of the point* $\check{c}_0$ *has a root* $\check{c}$.
2. *In the neighborhood of the zeroth approximation* $\check{c}_0$ *there are inequalities*

$$\|J_k^+\| \leq \sigma_1(k), \ \ \|d^2\varphi(\xi_k; \check{c} - \check{c}_k)\| \leq \sigma_2(k) \cdot \|\check{c} - \check{c}_k\|.$$

3. *The following constant exists*

$$\theta := \sup_{k\in\mathbb{N}} \Big\{ \frac{\sigma_1(k)\sigma_2(k)}{2} \Big\}.$$

*Then, under conditions*

$$P_{J_k^*} = 0, \ \ J_k := \varphi'(\check{c}_k) \in \mathbb{R}^{n\times(2r+\omega)}, \ \ \theta \cdot \|\check{c} - \check{c}_0\| < 1 \tag{5}$$

*to find the solution* $\check{c}$ *of equation (4) the iteration scheme (6)*

$$\check{c}_{k+1} = \check{c}_k - J_k^+\varphi(\check{c}_k), \ \ k \in \mathbb{N}, \tag{6}$$

*is applicable, and the rate of convergence of the sequence* $\check{c}$ *of equation (4) is quadratic.*

Here $P_{J_k^*} : \mathbb{R}^m \to \mathbb{N}(J_k^*)$ is an orthogonal projector of the matrix $J_k$ and $J_k^+$ is the pseudoinverse Moore–Penrose matrix [1]. Note that condition (5) is equivalent [5, 7, 8] to the requirement of completeness of the rank matrix $J_k$ and is possible only in case $m \leq n$.

The results of the proved Theorem 1 were obtained making no use of the central canonical form, perfect pairs, and matrix triplets [1, 2].

The proposed scheme of studies of the nonsingular integral-differential boundary value problem (1) can be transferred analogously to [1, 4, 6, 11] onto nonlinear singular integral-differential boundary value problem. On the other hand, in the case of nonsolvability, the nonsingular integral-differential boundary value problems can be regularized analogously [3, 14].

Conditions for the solvability of the linear boundary-value problem for systems of differential-algebraic equations with the variable rank of the leading-coefficient matrix and the corresponding solution construction procedure have been found in the paper [9].

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