# The Asymptotic Properties of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of Second Order Differential Equations with the Product of a Regularly Varying Function of Unknown Function and a Rapidly Varying Function of its First Derivative

**O. O. Chepok**

*Southern Ukrainian National Pedagogical University K. D. Ushinsky, Odesa, Ukraine*
*E-mail:* olachepok@ukr.net

We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y')\varphi_1(y). \tag{1}$$

In this equation $\alpha_0 \in \{-1; 1\}$, functions $p : [a, \omega[ \to ]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to ]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, $\Delta_{Y_i}$ is either the interval $[y_i^0, Y_i[$ or the interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$, we put $y_i^0 > 0$ $(y_i^0 < 0)$.

We also suppose that function $\varphi_1$ is a regularly varying as $y \to Y_1$ function of index $\sigma_1$ [7, p. 10-15], function $\varphi_0$ is twice continuously differentiable on $\Delta_{Y_0}$ and satisfies the next conditions

$$\varphi_0'(y) \ne 0 \text{ as } y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function $\varphi_0$ and its derivative of the first order are rapidly varying functions as the argument tends to $Y_0$ [1]. Thus, the investigated differential equation contains the product of a regularly varying function of unknown function and a rapidly varying function of its first derivative in its right-hand side.

Previously we obtained results for this kind of equation containing a rapidly varying function of unknown function and a regularly varying function of its first derivative [2].

The main aim of the article is the investigation of conditions of the existence of following class of solutions of equation (1).

**Definition 1.** The solution $y$ of equation (1), defined on the interval $[t_0, \omega[ \subset [a, \omega[$, is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$, if the following conditions take place

$$y^{(i)} : [t_0, \omega[ \to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

This class of solutions was defined in the work by V. M. Evtukhov [3] for the $n$-th order differential equations of Emden–Fowler type and was concretized for the second-order equation. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions [4], every such solution belongs to one of four non-intersecting sets according to the value of $\lambda_0 : \lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\lambda_0 = 0$, $\lambda_0 = 1$, $\lambda_0 = \pm\infty$.

Now we consider the case $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$ of such solutions, every $P_\omega(Y_0, Y_1, \lambda_0)$-solution and its derivative satisfy the following limit relations

$$\frac{y'(t)}{y(t)} = \frac{\lambda_0}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)], \quad \frac{y''(t)}{y'(t)} = \frac{1}{(\lambda_0 - 1)\pi_\omega(t)}\,[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{3}$$

From conditions (3) it follows that such $P_\omega(Y_0, Y_1, \lambda_0)$-solutions are regularly varying functions of index $\frac{\lambda_0}{\lambda_0 - 1}$, and their derivatives are regularly varying functions of index $\frac{1}{\lambda_0 - 1}$ as $t \uparrow \omega$ [7].

To formulate the main result, we introduce the following definitions.

**Definition 2.** Let $Y \in \{0, \infty\}$, $\Delta_Y$ is some one-sided neighborhood of $Y$. Continuous-differentiable function $L : \Delta_Y \to ]0; +\infty[$ is called ( [6], p.2-3) a normalized slowly varying function as $z \to Y$ $(z \in \Delta_Y)$ if the next statement is valid

$$\lim_{\substack{y \to Y \\ y \in \Delta_Y}} \frac{yL'(y)}{L(y)} = 0.$$

**Definition 3.** We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to ]0; +\infty[$ satisfies the condition $S$ as $z \to Y$, if for any continuous differentiable normalized slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L : \Delta_{Y_i} \to ]0; +\infty[$ the next relation is valid

$$\theta(zL(z)) = \theta(z)(1 + o(1)) \ \text{ as } \ z \to Y, \ \ z \in \Delta_Y.$$

**Definition 4.** We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $L_0 : \Delta_Y \to ]0; +\infty[$ satisfies the condition $S_1$ as $z \to Y$ if for any finite segment $[a; b] \subset ]0; +\infty[$ the next inequality is true

$$\limsup_{\substack{z \to Y \\ z \in \Delta_Y}} \Big| \ln|z| \cdot \Big(\frac{L(\lambda z)}{L(z)} - 1\Big) \Big| < +\infty \ \text{ for all } \ \lambda \in [a; b].$$

Conditions $S$ and $S_1$ are satisfied by functions $\ln|y|$, $|\ln|y||^\mu$ $(\mu \in R)$, $\ln|\ln|y||$ and many others.

Introduce the necessary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t - \omega & \text{as } \omega < +\infty, \end{cases} \quad \theta_1(y) = \varphi_1(y)|y|^{-\sigma_1},$$

$$\Phi_0(z) = \int_{A_\omega}^{z} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)}\,, \quad A_\omega = \begin{cases} y_1^0 & \text{as } \displaystyle\int_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = \pm\infty, \\ Y_1 & \text{as } \displaystyle\int_{y_1^0}^{Y_1} \frac{ds}{|s|^{\sigma_1}\varphi_0(s)} = const, \end{cases}$$

$$Z_0 = \lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_1}}} \Phi_0(z), \quad \Phi_1(z) = \int_{A_\omega}^{z} \Phi_0(s)\,ds, \quad Z_1 = \lim_{\substack{z \to Y_1 \\ z \in \Delta_{Y_1}}} \Phi_1(z),$$

$$F(t) = \frac{\pi_\omega(t) I_1'(t)}{\Phi_1^{-1}(I_1(t))\Phi_1'(\Phi_1^{-1}(I_1(t)))}\,,$$

and in the case

$$y_0^0 \lim_{t \uparrow \omega} |\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0 - 1}} = Y_0,$$

we have

$$I(t)=\alpha_0 y_0^0\cdot\Big|\frac{\lambda_0-1}{\lambda_0}\Big|^{\sigma_1}\cdot\int\limits_{B_\omega^0}^{t}|\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau,$$

$$B_\omega^0=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega |\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega |\pi_\omega(\tau)|^{\sigma_1}p(\tau)\theta_1\big(|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}y_0^0\big)\,d\tau<+\infty,\end{cases}$$

$$I_1(t)=\int\limits_{B_\omega^1}^{t}\frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau,\quad B_\omega^1=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau=\pm\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega \frac{I(\tau)\Phi_0^{-1}(I(\tau))}{(\lambda_0-1)\pi_\omega(\tau)}\,d\tau<+\infty,\end{cases}$$

where $b\in[a;\omega[$ is chosen so that

$$y_0^0\lim_{t\uparrow\omega}|\pi_\omega(\tau)|^{\frac{\lambda_0}{\lambda_0-1}}\in\Delta_{Y_0}\ \text{ as }\ t\in[b;\omega].$$

**Note 1.** From conditions (3) of the function $\varphi_0$ it follows that $Z_0,Z_1\in\{0,+\infty\}$ and

$$\lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}}\frac{\Phi_0''(z)\cdot\Phi_0(z)}{(\Phi_0'(z))^2}=1,\quad \lim_{\substack{z\to Y_1\\ z\in\Delta_{Y_1}}}\frac{\Phi_1''(z)\cdot\Phi_1(z)}{(\Phi_1'(z))^2}=1. \tag{4}$$

**Note 2.** The following statements are true:

1)

$$\Phi_0(z)=(\sigma_1-1)\frac{\varphi_0^{\frac{\sigma_1}{\sigma_1-1}}(z)}{\varphi_0'(z)}\,[1+o(1)]\ \text{ as }\ z\to Y_1,\ \ y\in\Delta_{Y_1}.$$

Hence we have

$$\operatorname{sign}(\varphi_0'(z)\Phi_0(z))=\operatorname{sign}(\sigma_1-1)\ \text{ as }\ z\in\Delta_{Y_1}.$$

2)

$$\Phi_1(z)=\frac{\Phi_0^2(z)}{y\Phi_0'(z)}\,[1+o(1)]\ \text{ as }\ z\to Y_1,\ \ z\in\Delta_{Y_1}.$$

Hence we have

$$\operatorname{sign}(\Phi_1(z))=y_0^0\ \text{ as }\ z\in\Delta_{Y_1}.$$

3) The functions $\Phi_0^{-1}$ and $\Phi_1^{-1}$ exist and are slowly varying functions as inverse to rapidly varying functions as the arguments tend to $Y_1$ functions.

4) The function $\Phi_1'(\Phi_1^{-1})$ is a regularly varying function of the index 1 as the argument tends to $Y_1$.

Indeed, from (4) we have

$$\lim_{z\to Z_1}\frac{(\Phi_1'(\Phi_1^{-1}(z)))'z}{\Phi_1'(\Phi_1^{-1}(z))}=\lim_{z\to Z_1}\frac{\Phi_1''(\Phi_1^{-1}(z))z}{(\Phi_1'(\Phi_1^{-1}(z)))^2}$$
$$=\lim_{y\to Y_1}\frac{\Phi_1''(\Phi_1^{-1}(\Phi_1(y)))\Phi_1(y)}{(\Phi_1'(\Phi_1^{-1}(\Phi_1(y))))^2}=\lim_{y\to Y_1}\frac{\Phi_1''(y)\Phi_1(y)}{(\Phi_1'(y))^2}=1.$$

Let $Y\in\{0,\infty\}$, $\Delta_Y$ be some one-sided neighborhood of $Y$. A continuous-differentiable function $L:\Delta_Y\to]0;+\infty[$ is called [6, p. 2-3] a normalized slowly varying function as $z\to Y$ $(z\in\Delta_Y)$ if the next statement is valid

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0. \tag{5}$$

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition $S$ as $z\to Y$, if for any normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0;+\infty[$ the following equality takes place as $z\to Y$ $(z\in\Delta_Y)$,

$$\theta(zL(z))=\theta(z)(1+o(1)).$$

We will consider that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L_0:\Delta_Y\to]0;+\infty[$ satisfies the condition $S_1$ as $z\to Y$ if for any finite segment $[a;b]\subset]0;+\infty[$ the next inequality is true

$$\limsup_{\substack{z\to Y\\ z\in\Delta_Y}}\Big|\ln|z|\cdot\Big(\frac{L(\lambda z)}{L(z)}-1\Big)\Big|<+\infty\ \text{ for all }\ \lambda\in[a;b].$$

Conditions $S$ and $S_1$ are satisfied by functions $\ln|y|$, $|\ln|y||^{\mu}$ $(\mu\in R)$, $\ln|\ln|y||$ and many others.

The following theorem takes place.

**Theorem.** *Let $\sigma_1\in\mathbb{R}\setminus\{1\}$, the function $\theta_1$ satisfy the condition $S$, and the functions $\theta_1$ and $\Phi_1^{-1}\cdot\frac{\Phi_1'}{\Phi_1}(\Phi_1^{-1})$ satisfy the condition $S_1$. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, in case $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, it is necessary, and if the following condition takes place*

$$I(t)I_1(t)\sigma_1(\lambda_0-1)<0\ \ \text{if}\ \ t\in]b,\omega[\,,$$

*and there is a finite or infinite limit*

$$\frac{\sqrt{|\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}|}}{\ln|I_1(t)|}\,,$$

*it is sufficient that the next conditions*

$$\pi_\omega(t)y_1^0y_0^0\lambda_0(\lambda_0-1)>0,\quad y_1^0\alpha_0(\lambda_0-1)\pi_\omega(t)>0\ \ \text{as}\ \ t\in[a;\omega[\,,$$
$$y_0^0\cdot\lim_{t\uparrow\omega}|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}I_1(t)=Z_1,$$
$$\lim_{t\uparrow\omega}\frac{I'(t)I_1(t)}{I_1'(t)I(t)}=1,\quad \lim_{t\uparrow\omega}\frac{\Phi(\Phi_1^{-1}(I_1(t)))}{I(t)}=1,\quad \lim_{t\uparrow\omega}F(t)=\frac{1}{\lambda_0-1}$$

*are fulfilled. Moreover, for each such solution the next asymptotic representations as $t\uparrow\omega$ take place*

$$y'(t)=\Phi_1^{-1}(I_1(t))[1+o(1)],\quad y(t)=\frac{(\lambda_0-1)\Phi_1^{-1}(I_1(t))\pi_\omega(t)}{\lambda_0}\,[1+o(1)].$$

For the equation under the investigation the question of the active existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions, in case $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, that have the received asymptotic representations, has been reduced to the question of the existence of infinitely small as arguments tend to $\omega$ solutions of the corresponding, equivalent to the investigated equation, systems of non-autonomous quasi-linear differential equations that admit applications of the known results from the works by V. M. Evtukhov and A. M. Samoilenko [5].

## References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation.* Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1987.

[2] O. O. Chepok, Asymptotic representations of a class of regularly varying solutions of differential equations of the second order with rapidly and regularly varying nonlinearities. *Mem. Differ. Equ. Math. Phys.* **74** (2018), 79–92.

[3] V. M. Evtukhov, Asymptotic representations of solutions of non-autonomous ordinary differential equations. *Diss. Dr. Phys.-Math. Sciences: 01.01.02 – Differential equations*, Odessa I. I. Mechnikov National University, Odessa, Ukraine, 1997.

[4] V. M. Evtukhov, Asymptotic properties of solutions of $n$-order differential equations. (Russian) *Ukrainian Mathematics Congress – 2001 (Ukrainian)*, 15–33, *Natsīonal. Akad. Nauk Ukraïni, ĪInst. Mat., Kiev*, 2002.

[5] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. (Russian) *Ukr. Mat. Zh.* **62** (2010), no. 1, 52–80; translation in *Ukr. Math. J.* **62** (2010), no. 1, 56–86.

[6] V. Marić, *Regular Variation and Differential Equations.* Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.

[7] E. Seneta, *Regularly Varying Functions.* Lecture Notes in Mathematics, Vol. 508. Springer-Verlag, Berlin–New York, 1976.