On the Solvability of a Three Point Boundary Value Problem for First Order Linear Functional Differential Equations

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We consider a boundary value problem

$$\dot{x}(t) = (T^+x)(t) - (T^-x)(t) + f(t), \quad t \in [0,1], \tag{1}$$

$$x(0) + x(1) = 2x(c), \tag{2}$$

where $c \in (0,1)$ is a given point, T^+ and T^- are linear *positive* operators, acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ with the standard norms, $f \in \mathbf{L}[0,1]$ (here positive operators map continuous non-negative functions into non-negative integrable functions). An absolutely continuous function $x : [0,1] \to \mathbb{R}$ is called a solution of problem (1), (2) if it satisfies equation (1) for almost all $t \in [0,1]$ and satisfies three point boundary value condition (2).

If we put $c = 0$ or $c = 1$, then condition (2) coincides with the periodic boundary value condition. Integral necessary and sufficient conditions for the unique solvability of the periodic boundary value problem for equation (1) in terms of two quantities $\int\limits_0^1 (T^+\mathbf{1})(s)\,ds$ and $\int\limits_0^1 (T^-\mathbf{1})(s)\,ds$ are known [1] (here $\mathbf{1} : [0,1] \to \mathbb{R}$ is the unit function). We formulate these conditions in the following form.

Theorem 1 ([1]). *Let two nonnegative numbers $\mathcal{T}^+$ and $\mathcal{T}^-$ be given. The periodic boundary value problem for equation* (1) *is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities*

$$\int\limits_0^1 (T^+\mathbf{1})(s)\,ds = \mathcal{T}^+, \quad \int\limits_0^1 (T^-\mathbf{1})(s)\,ds = \mathcal{T}^-,$$

if and only if the inequalities

$$\frac{X}{1-X} < Y < 2\Big(1 + \sqrt{1-X}\Big), \quad X < 1,$$

are fulfilled, where

$$X = \min\{\mathcal{T}^+, \mathcal{T}^-\}, \quad Y = \max\{\mathcal{T}^+, \mathcal{T}^-\}.$$

Various three-point boundary value problems are also considered for functional differential equations (see, for example, [2]). Similar integral necessary and sufficient conditions for three point problems, in particular, problem (1), (2), as far as we know, have not yet been obtained. It is natural to consider the conditions for the unique solvability of this problem in terms of four parameters, namely, the integrals of $T^+\mathbf{1}$ and $T^-\mathbf{1}$ over intervals $[0,c]$ and $[c,1]$:

$$\int\limits_0^c (T^+\mathbf{1})(s)\,ds \equiv P_L, \quad \int\limits_c^1 (T^+\mathbf{1})(s)\,ds \equiv P_R, \tag{3}$$

$$\int\limits_0^c (T^-\mathbf{1})(s)\,ds \equiv M_L, \quad \int\limits_c^1 (T^-\mathbf{1})(s)\,ds \equiv M_R. \tag{4}$$

We are interested in the structure of the uniquely solvable set $\Omega \in \mathbb{R}^4$, that is, the set of all points (P_L, P_R, M_L, M_R) for which any problem (1), (2) with positive linear operators T^+, T^- satisfying equalities (3), (4) is uniquely solvable. These conditions of solvability (necessary and sufficient) turn out to be rather complicated.

Define

$$\Delta \equiv P_L + M_R - P_R - M_L, \quad P \equiv P_L + P_R, \quad M \equiv M_L + M_R.$$

Theorem 2. *Let four non-negative numbers P_L, P_R, M_L, M_R be given, $\Delta > 0$. Then the boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if the following conditions are fulfilled:*

$$\Delta > M_L P_L + M_R P_R + 2M_L M_R; \quad \Delta > M_L P_L + M_R P_R + 2P_L P_R;$$
$$P_R + M_L < 1; \quad \Delta > \frac{(P_L + M_R)^2}{4};$$
$$\Delta + s_p^2 - s_p(P + M_R) > 0,$$
$$\Delta + s_p^2 - s_p(P + 2M_R) + 2M_R P_R + M_R M_L > 0,$$
$$\Delta + s_p^2 - s_p(P + 2M_R) + 2M P_R > 0 \ \text{ for all } \ s_p \in [P_R, P];$$
$$\Delta + s_m^2 - s_m(P_L + M) > 0,$$
$$\Delta + s_m^2 - s_m(2P_L + M) + 2P_L M_L + P_R P_L > 0,$$
$$\Delta + s_m^2 - s_m(2P_L + M) + 2P\, M_L > 0 \ \text{ for all } \ s_m \in [M_L, M].$$

Some conditions of the previous theorem can be simplified. Define several values

$$r_1 \equiv (P_L + P_R)M_R + \frac{(P_L + P_R - M_R)_+^2}{4},$$
$$r_2 \equiv (M_L + M_R)P_L + \frac{(M_L + M_R - P_L)_+^2}{4},$$
$$r_3 \equiv 2P_L M_R + \frac{(P_L + P_R - 2M_R)_+^2}{4} - M_L \min\{M_R, 2P_R\},$$
$$r_4 \equiv 2P_L M_R + \frac{(M_L + M_R - 2P_L)_+^2}{4} - P_R \min\{P_L, 2M_L\},$$
$$r_5 \equiv M_L P_L + M_R P_R + 2\max\{M_L M_R, P_L P_R\},$$

where $(a)_+ = \max\{0, a\}$ for every $a \in \mathbb{R}$. The numbers r_i' can be obtained from the numbers r_i, respectively, when replacing P_L with M_R, P_R with M_L, M_L with P_R, M_R with P_L; $\Delta' \equiv -\Delta$.

Theorem 3. *Let four non-negative numbers P_L, P_R, M_L, M_R be given. Then the boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if*

$$\Delta > 0, \quad \Delta > \max\big\{r_1, r_2, r_3, r_4, r_5\big\},$$

or

$$\Delta' > 0, \quad \Delta' > \max\big\{r_1', r_2', r_3', r_4', r_5'\big\}.$$

Corollary 1. *If the conditions of Theorem 3 are satisfied, then*

$$P_R + M_L < 1, \quad P_L + M_R < 2\Big(1 + \sqrt{1 - P_R - M_L}\,\Big).$$

or

$$P_L + M_R < 1, \quad P_R + M_L < 2\Big(1 + \sqrt{1 - P_L - M_R}\,\Big).$$

Corollary 2. *The uniquely solvable set Ω is non-empty and consists of two path-connected components, Ω_+, for whose points $\Delta > 0$, and Ω_-, for whose points $\Delta < 0$. For each point of each component of the uniquely solvable set, the intersection of this component with each straight line that is parallel to one of the axes and passes through this point is an open interval or a half-open interval of the form $[0, d)$.*

Thus, Corollary 2 gives a good idea of the structure of the uniquely solvable set. It remains only to study the boundaries of the set in more detail.

If $P_L = 0$ or $M_R = 0$, then the conditions for the solvability turn out to be simple (and close to the conditions for the solvability of the periodic boundary value problem).

Theorem 4. *Let $M_R = 0$ and non-negative numbers P_L, P_R, M_L be given. Suppose*

$$P_L > P_R + M_L.$$

Then problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if

$$\frac{P_R + M_L}{1 - 2P_R - M_L} < P_L < 2 - P_R + 2\sqrt{1 - 2P_R - M_L}\,, \quad 2P_R + M_L < 1.$$

Let $P_L = 0$ and non-negative numbers M_R, P_R, M_L be given. Suppose

$$M_R > P_R + M_L.$$

Then problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying equalities (3), (4) if and only if

$$\frac{P_R + M_L}{1 - P_R - 2M_L} < M_R < 2 - P_R + 2\sqrt{1 - P_R - 2M_L}\,, \quad P_R + 2M_L < 1.$$

It is also easy to construct the set of unique solvability for two zero parameters $M_L = 0$ and $P_R = 0$. This section plays an important role in the construction of the entire solvability set (one should pay attention to Corollary 3 below).

Theorem 5. *Let $M_L = 0$ and $P_R = 0$. Let non-negative numbers P_L, M_R be given. The boundary value problem (1), (2) is uniquely solvable for all linear positive operators T^+, T^- satisfying the equalities (3), (4) if and only if*

$$P_L \in (0, 4), \quad M_R = 0, \tag{5}$$

$$0 \le P_L < \begin{cases} 2\Big(1 - M_R + \sqrt{1 - M_R}\,\Big), & M_R \in \Big(0, \dfrac{3}{4}\Big), \\ \dfrac{M_R}{2M_R - 1}\,, & M_R \in \Big[\dfrac{3}{4}, \dfrac{3}{2}\Big), \\ \dfrac{1 - M_R + \sqrt{2M_R + 1}}{2}\,, & M_R \in \Big[\dfrac{3}{2}, 4\Big). \end{cases} \tag{6}$$

Let $\Omega^{0,0} = \{(t, s) : \ (P_L, 0, 0, M_R) \in \Omega_+\}$ be the set of all points (P_L, M_R) satisfying (5), (6). It is the section of the solvability set Ω_+ for $P_R = 0$, $M_L = 0$. Constructing a section of the solvability set when only one of the numbers P_R and M_L is zero is not such an easy task.

The components Ω_+ and Ω_- of the solvability set are symmetric. We investigate the set Ω_+, that is, the case when

$$\Delta = P_L + M_R - P_R - M_L > 0.$$

It turns out that the relations defining the boundaries of the solvability set in this case are easily resolved with respect to "small" parameters $P_R \equiv t$, $M_L \equiv s$, and have the form

$$\begin{aligned} s &< P_L + M_R - P_L M_R - t(1+M_R) - \frac{(P_L + t - M_R)_+^2}{4}\,, \\ \big(1 - \min\{M_R, 2t\}\big)s &< P_L + M_R - t - 2P_L M_R - \frac{(P_L + t - 2M_R)_+^2}{4}\,, \\ (1+P_L)s &< P_L + M_R - t(1 + M_R + 2P_L), \end{aligned} \tag{7}$$

and

$$\begin{aligned} t &< P_L + M_R - P_L M_R - s(1+P_L) - \frac{(M_R - P_L + s)_+^2}{4}\,, \\ \big(1 - \min\{P_L, 2s\}\big)t &< P_L + M_R - 2P_L M_R - s - \frac{(M_R - 2P_L + s)_+^2}{4}\,, \\ (1+M_R)t &< P_L + M_R - s(1 + 2M_R + P_L). \end{aligned} \tag{8}$$

Let $(P_L, P_R = t, M_L = s, M_R) \in \Omega_+$. Therefore, inequalities (7), (8) are fulfilled. Then it is easy to show that $(P_L, 0, 0, M_R) \in \Omega_+$.

Let sections of Ω_+ be defined by

$$\Omega_{P_L, M_R} = \big\{(t,s) :\ (P_L, t, s, M_R) \in \Omega_+\big\}.$$

Corollary 3. *The set Ω_{P_L,M_R} is not empty if and only if $(P_L, M_R) \in \Omega^{0,0}$.*

Corollary 4. *Let $(P_R, M_L) \in \Omega_{P_L,M_R}$. Then $(t,s) \in \Omega_{P_L,M_R}$ for all $t \in [0, P_R]$, $s \in [0, M_L]$.*

For some pairs of P_L and M_R, the border of Ω_{P_L,M_R} is relatively simple. Due to symmetry, it is sufficient to consider the case $P_L \le M_R$.

Theorem 6. *Let $M_R \in [0,1]$, $P_L \le \max\{M_R, 1/2\}$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - s(1+P_L)}{1 + M_R + 2P_L}\Big),\ s \in \Big[0, \frac{P_L}{1 + M_R + P_L}\Big]; \\ & t \in \Big[0, \frac{P_L + M_R - s(1 + 2M_R + P_L)}{1 + M_R}\Big),\ s \in \Big[\frac{P_L}{1 + M_R + P_L}, \frac{P_L + M_R}{1 + P_L}\Big]\bigg\}. \end{aligned}$$

Theorem 7. *Let $M_R \in [1, 1+\sqrt{2})$, $P_L \in [0, (1 + \sqrt{2} - M_R)^2/4)$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - 2P_L M_R - s}{1 + M_R}\Big),\ s \in \Big[0, \frac{P_L M_R}{2M_R + P_L}\Big]; \\ & t \in \Big[0, \frac{P_L + M_R - s(1 + 2M_R + P_L)}{1 + M_R}\Big),\ s \in \Big[\frac{P_L M_R}{2M_R + P_L}, \frac{P_L + M_R}{1 + 2M_R + P_L}\Big]\bigg\}. \end{aligned}$$

Theorem 8. *Let $(M_L, P_R) \in \Omega^{0,0}$, $M_L \in [8/11, 4)$, $2(4 - M_R)/9 \le P_L \le M_R$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & 0 \le t < \frac{P_L + M_R - 2P_L M_R - s - (M_R - 2P_L + s)^2/4}{1 - 2s}\,, \\ & s \in \Big[0, 2\sqrt{(1-P_L)(1+2M_R)} - M_R - 2 + 2P_L\Big)\bigg\}. \end{aligned}$$

Theorem 9. *Let $M_L \in [3,4)$, $P_L \le (1 - M_R + \sqrt{2M_R + 1})/4$. Then*

$$\begin{aligned} \Omega_{P_L,M_R} = \bigg\{(t,s) :\ & t \in \Big[0, \frac{P_L + M_R - 2P_L M_R - s - (M_R - P_L + s)^2/4}{1 - \min\{P_L, 2s\}}\Big), \\ & s \in \Big[0, 2\sqrt{(1-P_L)(1+2M_R)} - M_R - 2 + 2P_L\Big)\bigg\}. \end{aligned}$$

Acknowledgements

The work is performed as part of the State Task of the Ministry of Education and Science of the Russian Federation (project # 1.5336.2017/8.9).

References

[1] R. Hakl, A. Lomtatidze and B. Půža, On periodic solutions of first order linear functional differential equations. *Nonlinear Anal.* **49** (2002), no. 7, Ser. A: Theory Methods, 929–945.

[2] A. Rontó and M. Rontó, On nonseparated three-point boundary value problems for linear functional differential equations. *Abstr. Appl. Anal.* **2011**, Art. ID 326052, 22 pp.