Existence of a Complete Unstable Differential System with Perron and Upper-Limit Partial Stability

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The present report deals with a recently introduced [7] concept of the Qualitative Theory of Differential Equations, namely the Perron stability. It continues the series of papers by the author [1] and [2], reinforcing their results. The first of these works corrected the defect stated in Remark 4 to Theorem 1 [8], but the differential system constructed there possessed a non-zero (though limited on the whole semi-axis of time) linear approximation at zero. In the second paper the system with the same properties, but already with a zero linear approximation at zero, was constructed.

The following reinforcement of the above results consists in constructing a system with both Perron and upper-limit complete instability (and thus also Lyapunov global instability) and at the same time not just partial (as in all the examples discussed above) but even massive partial instability.

For a number $n \in \mathbb{N}$ and for a region $G \ni 0$ of the Euclidean space $\mathbb{R}^n$, consider the system

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}_+ \equiv [0,\infty), \ \ x \in G, \tag{1}$$

with the right hand side $f : \mathbb{R}_+ \times G \to \mathbb{R}^n$ satisfying the conditions

$$f, f'_x \in C(\mathbb{R}_+ \times G), \quad f(t,0) = 0, \ \ t \in \mathbb{R}_+, \tag{2}$$

and therefore admitting the *zero* solution. Let us denote by $\mathcal{S}_*(f)$ the set of all *non-continuable* non-zero solutions to system (1) and by $\mathcal{S}_\delta(f)$ – the subset in $\mathcal{S}_*(f)$ consisting of those and only those solutions x which satisfy the *initial condition* $|x(0)| < \delta$ (here $|\cdot|$ is the Euclidean norm in the space $\mathbb{R}^n$).

Definition 1. Let us say that for system (1) (more exactly, for its *zero* solution, which we will not mention further for brevity) the following *Perron property* takes place:

1) *Perron stability* if for any $\varepsilon > 0$ there exists such $\delta > 0$ that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the condition
$$\varliminf_{t\to+\infty} |x(t)| < \varepsilon; \tag{3}$$

2) *Perron instability* if there is no Perron stability, namely, if there exists such $\varepsilon > 0$ that for any $\delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ does not satisfy condition (3);

3) *complete Perron instability* if for some $\varepsilon, \delta > 0$ no solution $x \in \mathcal{S}_\delta(f)$ satisfies condition (3);

4) *particular Perron stability* if there is no complete Perron instability, namely, if for any $\varepsilon, \delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ satisfies condition (3);

5) *asymptotic Perron stability* if for some $\delta > 0$ any solution $x \in \mathcal{S}_\delta(f)$ satisfies condition

$$\varliminf_{t\to+\infty} |x(t)| = 0; \tag{4}$$

6) *asymptotic Perron instability* if there is no asymptotic Perron stability, namely, if for any $\delta > 0$ some solution $x \in \mathcal{S}_\delta(f)$ does not satisfy condition (4).

The definition of the Perron properties essentially relies on the transition to the lower limit when $t \to +\infty$ (see conditions (3) and (4) in Definition 1). Therefore they could also be called lower-limit ones and it would be appropriate to consider also their natural analogues using the upper limit instead of the lower limit. To do this, let us formulate

Definition 2. Let us compare to each Perron property from Definition 1 its upper-limit analogue, namely: *stability, instability, complete instability, particular stability, asymptotic stability, asymptotic instability* are obtained by repeating respectively the descriptions from steps 1–6 of Definition 1 with replacement in them conditions (3) and (4) by conditions

$$\varlimsup_{t\to+\infty} |x(t)| < \varepsilon \tag{5}$$

and, respectively,

$$\varlimsup_{t\to+\infty} |x(t)| = 0. \tag{6}$$

Let us emphasize that in Definition 1 of the Perron properties conditions (3) and (4), as well as in Definition 2 of the upper-limit properties, conditions (5) and (6) are considered as not fulfilled, in particular already in the case when the solution x is simply not defined on the whole semi-axis $\mathbb{R}_+$, which takes place if and only if its corresponding phase curve reaches the limit of the phase region G in finite time (according to the solution continuity theorem; see, for example, Theorem 23 [9]). Each Perron and upper-limit property 2–5 of Definitions 1 and 2 according to Theorem 3 from [10] is

(a) *local in its initial value*, i.e. to establish it is sufficient for an arbitrary fixed value of $r > 0$ to consider those and only those solutions x which satisfy the condition $|x(0)| < r$;

(b) *local in the phase variable*, i.e. to establishit is sufficient for an arbitrary fixed value of $r > 0$ to know the values of each solution x at those and only those moments $t \in \mathbb{R}_+$ for which it satisfies the condition $|x(t)| < r$.

Therefore to complete the picture it seems appropriate also to consider the properties characterizing the behaviour not only of near-zero solutions but of all solutions in general, i.e. having, so to speak, a global character. For this purpose let us formulate

Definition 3. Let us consider that the following property for system (1) takes place:

1) *Perron* or *upper-limit global stability* if any of its solutions $x \in \mathcal{S}_*(f)$ satisfies conditions (4) or (6), respectively;

2) *Perron* or *upper-limit partial instability* if it does not possess Perron or upper-limit global stability, namely, there is at least one its solution $x \in \mathcal{S}_*(f)$ that does not satisfy conditions (4) or (6), respectively;

3) *Perron* or *upper-limit partial stability* if for any $\varepsilon > 0$ at least one of its solution $x \in \mathcal{S}_*(f)$ satisfies, conditions (3) or (5), respectively;

4) *Perron* or *upper-limit global instability* if it does not possess Perron or upper-limit partial stability, namely, for some $\varepsilon > 0$ none of its solutions $x \in \mathcal{S}_*(f)$ satisfies conditions (3) or (5), respectively.

The main result of this paper is to prove the existence of the differential system, all the near-zero solutions of which tend to infinity at $t \to +\infty$ (hence the system is completely unstable both Perron and upper-limit) and all other solutions tend to zero. This means that such system is not globally unstable (neither Perron nor upper-limit) but it does possess both types of partial stability and not just partial one, but even massive partial stability (i.e. the condition of satisfying one of conditions (3) or (5) is imposed on the set of solutions of system (1)).

Theorem. *When $n = 2$, there exists system* (1) *satisfying conditions* (2) *and posessing the following three properties:*

1) *the right hand side of systen* (1) *is infinitely differentiable and*

$$f'_x(t,0) = 0, \quad t \in \mathbb{R}_+;$$

2) *for each solution x to system* (1)*, satisfying initial conditions $0 < |x(0)| < 1$ or $x(0) = (1,0)^T$ and also $|x(0)| = 1$ and $x_2(0) < 0$, there exists the equality*

$$\lim_{t\to+\infty} |x(t)| = +\infty;$$

3) *for all other solutions x to system* (1)*, satisfying initial conditions $|x(0)| > 1$ or $x(0) = (-1,0)^T$ and also $|x(0)| = 1$ or $x_2(0) > 0$, there exists the equality*

$$\lim_{t\to+\infty} |x(t)| = 0.$$

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