Sixth Order Accuracy Difference Schemes for the Helmholtz-Type Equation

Givi Berikelashvili[1,2]

[1]*A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University Tbilisi, Georgia*
[2]*Department of Mathematics, Georgian Technical University, Tbilisi, Georgia*
E-mails: bergi@rmi.ge, g.berikelashvili@gtu.ge

Nikoloz Kachakhidze

Department of Mathematics, Georgian Technical University, Tbilisi, Georgia
E-mail: N.Kachakhidze@gtu.ge

The first difference schemes with sixth order accuracy for approximation of elliptic equations were offered by S. Mikeladze [6, 7], and further were being studied by a number of authors. Convergence of these schemes with rate $O(h^6)$ were stated under condition that the solution of the differential problem belongs to the class $C^8(\overline{\Omega})$.

One of the most frequently encountering equations of numerical weather prediction, and fluid dynamics generally, is the Helmholtz-type diagnostic equation [5]. Below, we propose and investigate difference schemes approximating the following problem

$$\Delta u-\lambda u=f(x),\ \ x\in\Omega,\quad u(x)=0,\ \ x\in\Gamma, \tag{1}$$

where $\lambda\geq 0$ is a constant and $\Omega=\{x=(x_1,x_2):\ 0<x_\alpha<l,\ \alpha=1,2\}$ is the square with boundary Γ.

In $\overline{\Omega}=\Omega\cup\Gamma$ we introduce a grid $\overline{\omega}=\overline{\omega}_1\times\overline{\omega}_2$, where

$$\overline{\omega}_\alpha=\Big\{x_\alpha=i_\alpha h:\ i_\alpha=0,1,\dots,N,\ h=\frac{l}{N}\Big\},\ \ \gamma=\overline{\omega}\setminus\omega.$$

Besides

$$\omega_\alpha=\overline{\omega}_\alpha\cap(0;l),\ \ \omega_\alpha^+=\overline{\omega}_\alpha\cap(0;l],\ \ \omega=\omega_1\times\omega_2,\ \ \omega^+=\omega_1^+\times\omega_2^+,$$
$$\omega_{(1)}=\omega_1^+\times\omega_2,\ \ \omega_{(2)}=\omega_1\times\omega_2^+,\ \ \gamma=\overline{\omega}\setminus\omega.$$

Let

$$(y,v)_{\widetilde{\omega}}=\sum_{x\in\widetilde{\omega}}h^2y(x)v(x),\ \ \|y\|_{\widetilde{\omega}}^2=(y,y)_{\widetilde{\omega}}\ \text{ for }\ \widetilde{\omega}\subseteq\overline{\omega}.$$

Let's denote by H the set of grid functions given on $\overline{\omega}$ and vanishing on γ, with the scalar production and norm $(y,v)=(y,v)_\omega$, $\|y\|=\|y\|_\omega$.

Also, in space H, we introduce the norms

$$\|y\|_{(\alpha)}^2=(y,y)_{\omega(\alpha)},\ \ \alpha=1,2,$$
$$\|y\|_1^2=\|y\|_{W_2^1(\omega)}^2=\|y_{\overline{x}_1}\|_{(1)}^2+\|y_{\overline{x}_2}\|_{(2)}^2,$$
$$\|y\|_2^2=\|y\|_{W_2^2(\omega)}^2=\|y_{\overline{x}_1x_1}\|^2+\|y_{\overline{x}_2\overline{x}_2}\|^2+2\|y_{\overline{x}_1\overline{x}_2}\|_{\omega^+}^2,$$

It is supposed that
$$\|y\|_{W_2^0(\omega)} = \|y\|.$$
For functions with continuous argument we will use the following averaging operators
$$T_\alpha u = \int\limits_{-1}^{1} (1-|t|)u\big(x_1+(2-\alpha)th, x_2+(\alpha-1)th\big)\,dt, \ \ \alpha=1,2.$$
We will approximate problem (1) with the help of family of difference schemes dependent on a parameter ε:
$$Ay=\varphi(x), \ \ x\in\omega, \quad y(x)=0, \ \ x\in\gamma, \tag{2}$$
where
$$A \equiv \Big(1-\frac{\lambda^2h^4}{360}-(1-\varepsilon)\,\frac{\lambda h^2}{12}\,\Big(1-\frac{\lambda h^2}{12}\Big)\Big)(A_1+A_2) \\ -\frac{h^2}{6}\,\Big(1-\frac{\lambda h^2}{60}\,(7-5\varepsilon)\Big)A_1A_2+\lambda\Big(1+\frac{\lambda h^2}{12}\,\varepsilon\Big)E,$$
$$A_\alpha y\equiv -y_{\overline{x}_\alpha x_\alpha}, \ \ \alpha=1,2, \quad Ey\equiv y, \quad \varphi=\Big(1+\frac{\lambda h^2}{12}\,\varepsilon\Big)T_1T_2f+\frac{h^2}{240}\,(A_1A_2f+\lambda(A_1+A_2)f).$$
It can be proved that operator A is self-conjugate and positively defined in H; the following estimations
$$A\ge\delta(A_1+A_2)+\frac{4}{9}\,\lambda E, \ \ \delta\|y\|_1^2\le(Ay,y), \ \ \delta\|y\|_2\le\|Ay\|,$$
where
$$\delta=\frac{2}{3}\,\Big(1+\frac{\lambda h^2}{12}\,\varepsilon\Big),$$
are valid for it.

Positive definiteness of operator A ensures unique solvability of the difference scheme (2).

Substituting $y=z+u$ in (2), we get the problem
$$Az=\psi, \ \ x\in\omega, \quad z(x)=0, \ \ x\in\gamma \tag{3}$$
for error z, where $\psi=\varphi-Au$ is an approximation error.

Using equation (1) and the identity $T_\alpha\,\frac{\partial^2u}{\partial x_\alpha^2}=u_{\overline{x}_\alpha x_\alpha}$ we represent ψ in the form
$$\psi=\Big(1+\frac{\lambda h^2}{12}\,\varepsilon\Big)(A_1\eta_1+A_2\eta_2+A_3\eta_3)+\lambda\eta_4,$$
where
$$\eta_{3-\alpha}=T_\alpha u-u+\frac{h^2}{12}\,A_\alpha u-\frac{h^2}{240}\,A_\alpha\,\frac{\partial^2u}{\partial x_\alpha^2}\,, \ \ \alpha=1,2,$$
$$\eta_3=T_1T_2u-u+\frac{h^2}{12}\,(A_1+A_2)u-\frac{11}{720}\,h^4A_1A_2u-\frac{h^4}{240}\,(A_1+A_2)\Delta u,$$
$$\eta_4=\Big(A_1A_2\Delta u+\lambda(A_1+A_2)\Delta u+\frac{11}{3}\,\lambda A_1A_2u\Big)\,\frac{\varepsilon h^6}{12\cdot240}\,.$$
For the solution of problem (3) the following estimations are true:
$$\|z\|\le\frac{3}{2}\,\Big(\|\eta_1\|+\|\eta_2\|+\frac{\lambda l^2}{16}\,\big(\|\eta_3\|+\|\eta_4\|\big)\Big),$$
$$\|z\|_1\le\frac{3}{2}\,\Big(\|\eta_{1\overline{x}_1}\|_{(1)}+\|\eta_{2\overline{x}_2}\|_{(2)}+\frac{\lambda l}{4}\,\big(\|\eta_3\|+\|\eta_4\|\big)\Big),$$
$$\|z\|_2\le\frac{3}{2}\,\Big(\|\eta_{1\overline{x}_1x_1}\|_{(1)}+\|\eta_{2\overline{x}_2x_2}\|_{(2)}+\lambda\big(\|\eta_3\|+\|\eta_4\|\big)\Big).$$

It can be checked that expansions of linear (with respect to $u(x)$) functionals η_1, η_2, η_3, η_4 in the class of sufficiently smooth functions start from sixth order derivatives.

With the help of technique of investigation [1, 4, 8], based on using of approximating lemma of Bramble–Hilbert [2, 3], we become convinced in validness of the following

Theorem 1. *Let the solution of problem* (1) *belong to the space* $W_2^m(\Omega)$, $m > 3$. *Then the convergence of the difference scheme* (2) *at* $\varepsilon \geq 0$ *is characterized by the estimation*

$$\|y-u\|_{W_2^s(\omega)} \leq Mh^{m-s}\|u\|_{W_2^m}(\Omega), \;\; s=0,1,2, \;\; m\in(3,6+s].$$

References

[1] G. Berikelashvili, Construction and analysis of difference schemes for some elliptic problems, and consistent estimates of the rate of convergence. *Mem. Differential Equations Math. Phys.* **38** (2006), 1–131.

[2] J. H. Bramble and S. R. Hilbert, Bounds for a class of linear functionals with applications to Hermite interpolation. *Numer. Math.* **16** (1970/71), 362–369.

[3] T. Dupont and R. Scott, Polynomial approximation of functions in Sobolev spaces. *Math. Comp.* **34** (1980), no. 150, 441–463.

[4] B. S. Jovanović, *The Finite Difference Method for Boundary-Value Problems with Weak Solutions.* Posebna Izdanja [Special Editions], 16. Matematički Institut u Beogradu, Belgrade, 1993.

[5] J. A. Leslie, An inverse problem in super geometry. *Analysis on infinite-dimensional Lie groups and algebras (Marseille, 1997)*, 235–243, *World Sci. Publ., River Edge, NJ*, 1998.

[6] S. Mikeladze, Über die numerische Lösung der Differentialgleichung $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\varphi(x,y,z)$. (German) *C. R. (Dokl.) Acad. Sci. URSS, n. Ser.* **14** (1937), 177–179.

[7] S. E. Mikeladze, Über die numerische Lösung der Differentialgleichungen von Laplace und Poisson. (Russian) *Bull. Acad. Sci. URSS, Ser. Math.* **1938**, No. 2, 271–292.

[8] A. A. Samarskii, R. D. Lazarov and V. L. Makarov. *Difference Schemes for Differential Equations with Generalized Solutions.* Vysshaya Shkola, Moscow (1987).