Oscillation Problems for Generalized Emden–Fowler Equation

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Consider the super-linear generalized Emden–Fowler differential equation for $t \in I = [1, \infty)$

$$\big(|x'|^{\alpha} \operatorname{sgn} x'\big)' + b(t)g(x)|x|^{\beta} \operatorname{sgn} x = 0, \ \ 0 < \alpha < \beta, \tag{1}$$

and its special case

$$x'' + b(t) \ |x|^{\beta} \operatorname{sgn} x = 0, \ \ t \in I, \ \ \beta > 1, \tag{2}$$

where b is a positive absolute continuous function on I and g is a positive continuous function on $\mathbb{R}$.

A solution x of (1) is said to be *proper* if it is defined for all large t and $\sup\limits_{t \in [\tau, \infty)} |x(t)| > 0$ for any large τ. A proper solution x of (1) is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is said to be *nonoscillatory*.

It is well known, see, e.g., [8], that the coexistence of nontrivial oscillatory and nonoscillatory solutions is possible for (1). Further, in [10] the question as to whether oscillatory solutions of (2) may coexist with nonoscillatory ones having at least one zero has been posed and Kiguradze in [8] has negatively answered to this question.

Concerning the existence of nonoscillatory solutions, it is well known that the class $\mathbb{P}$ of all eventually positive solutions x of (1) can be divided into three subclasses, according to the asymptotic behavior of x as $t \to \infty$, namely

$$\begin{aligned}
\mathbb{M}^{+}_{\infty,\ell} &= \Big\{x \in \mathbb{P} : \lim_{t \to \infty} x(t) = \infty, \ \lim_{t \to \infty} x'(t) = \ell_x, \ 0 < \ell_x < \infty\Big\},\\
\mathbb{M}^{+}_{\infty,0} &= \Big\{x \in \mathbb{P} : \ \lim_{t \to \infty} x(t) = \infty, \ \lim_{t \to \infty} x'(t) = 0\Big\},\\
\mathbb{M}^{+}_{\ell,0} &= \Big\{x \in \mathbb{P} : \ \lim_{t \to \infty} x(t) = \ell_x, \ \lim_{t \to \infty} x'(t) = 0, \ 0 < \ell_x < \infty\Big\},
\end{aligned}$$

see, e.g., [4]. Solutions in $\mathbb{M}^{+}_{\infty,\ell}$, $\mathbb{M}^{+}_{\infty,0}$, $\mathbb{M}^{+}_{\ell,0}$ are called also *dominant solutions, intermediate solutions* and *subdominant solutions*, respectively. Such a terminology has been introduced by the Japanese mathematical school and it is due to the fact that, if $x \in \mathbb{M}^{+}_{\infty,\ell}$, $y \in \mathbb{M}^{+}_{\infty,0}$, $z \in \mathbb{M}^{+}_{\ell,0}$, then we have $x(t) > y(t) > z(t)$ for large t.

Necessary and sufficient conditions for the existence of subdominant and dominant solutions are easily available in the literature, see, e.g., [4] and the references therein. However, as far we know, until now no general necessary and sufficient conditions for existence of intermediate solutions of (2) are known; this fact mainly is due to the lack of sharp upper and lower bounds for intermediate solutions, see, e.g., [7, page 3], [9, page 2].

Another interesting problem which arises, is whether all three types of nonoscillatory solutions can simultaneously exist. This problem has a long history. For equation (2), it started sixty years ago by Moore–Nehari [10] in case $\beta > 1$ and Belohorec [3] in case $\beta < 1$. This study was continued in some papers by Kamo, Kusano, Naito, Tanigawa, Usami for the more general equation

$$\big(a(t)|x'|^{\alpha}\operatorname{sgn}x'\big)' + b(t)|x|^{\beta}\operatorname{sgn}x = 0, \tag{3}$$

where a is a positive continuous function, in both cases $\alpha = \beta$ and $\alpha \neq \beta$. In particular, under additional assumptions, it is proved that this triple coexistence is impossible, see [6] for more details. Finally, in [4] the study has been completed with a negative answer.

A much more subtle question concerns the possible coexistence between oscillatory solutions and nonoscillatory solutions. For the special case of (2) with $b(t) = 1/4$, that is for the equation

$$x'' + \frac{1}{4}\,t^{-(\beta+3)/2}|x|^{\beta}\operatorname{sgn}x = 0, \quad t \in I, \quad \beta > 1, \tag{4}$$

it has been proved in [10] that (4) has both oscillatory solutions and nonoscillatory ones. These nonoscillatory solutions are either subdominant solutions or intermediate solutions and both types exist. Moreover, intermediate solutions of (4) intersect the intermediate solution $\sqrt{t}$ infinitely many times.

Observe that, in view of [8, Theorem 8.5], equation (4) can be considered, roughly speaking, as the border equation between oscillation of at least one solution and nonoscillation of all solutions.

Our aim here is to present some results concerning the existence of oscillatory solutions and intermediate solutions for (1) and its special case (2). Moreover, we show also how the results in [10] for (4) concerning the coexistence between oscillatory solutions and intermediate solutions can be extended to the perturbed equation (2). These results are taken from [1, 2] and we refer these papers for more details.

Theorem 1. *Assume that $t^{\gamma}b(t)$ is nonincreasing on I, where $\gamma = (\alpha\beta + 2\alpha + 1)(\alpha + 1)^{-1}$ and*

$$\begin{gathered} g(u)\operatorname{sgn}u \text{ is nonincreasing on } (-\infty,0) \text{ and } (0,\infty);\ \lim_{u\to\infty} g(u) = M > 0, \\ \lim_{u\to\infty} g(u) = M > 0. \end{gathered} \tag{5}$$

If

$$\int\limits_1^{\infty} s^{\beta} b(s)\,ds = \infty,$$

then equation (1) *has infinitely many intermediate solutions.*

A necessary condition for the existence of intermediate solutions follows from the following oscillation result.

Theorem 2. *Assume* (5). *Then any solution of* (1) *is oscillatory if and only if*

$$\int\limits_1^{\infty}\bigg(\int\limits_t^{\infty} b(s)\,ds\bigg)^{1/\alpha}dt = \infty.$$

For equation (2) we have the coexistence of oscillatory solutions and intermediate solutions, as the following result shows.

Theorem 3. *Consider equation* (2) *with*

$$b(t) = t^{-(\beta+3)/2}c(t),$$

where c is a positive absolute continuous function on I. If

$$\lim_{t\to\infty} c(t) = c_0 > 0 \ \ \textit{and} \ \ \int_1^\infty |c'(t)|\,dt < \infty, \tag{6}$$

then we have:

(i$_1$) *Equation* (2) *has infinitely many oscillatory solutions. In addition, if $c'(t) \ge 0$, then every solution with zero is oscillatory.*

(i$_2$) *Equation* (2) *has infinitely many intermediate solutions x defined on I such that*

$$C_0 t^{1/2} \le x(t) \le C_1 t^{1/2} \ \ \textit{for large } t, \tag{7}$$

where C_0 and C_1 are suitable positive constants which does not depend on the choice of x. Moreover, intermediate solutions intersect the function

$$(4c(t))^{\frac{1}{1-\beta}} \sqrt{t}$$

infinitely many times.

If

$$\int_1^\infty a^{-1/\alpha}(s)\,ds = \infty,$$

Theorem 1 and Theorem 2 can be extended to the more general equation (3) using the change of the independent variable

$$s = A(t) + 1 - c, \quad X(s) = x(t), \ \ t \in [1,\infty), \ \ s \in [1,\infty),$$

see [1, Section 6] for more details. Moreover, Theorem 1 extends recent results in [4,5] and Theorem 2 shows that the oscillation property reads in the same way for (1) and the Emden–Fowler equation (1) with $g(t) \equiv 1$, see, e.g., [8, Chapter V].

For equation (2), Theorem 3(i$_3$) extends analogues results in [5, Theorem 2.1] and [1, Theorem 3.1], where b is required to be nonincreasing for $t \ge 1$.

The proof of Theorem 1 is mainly based on certain asymptotic property of a suitable associated energy function, see [1, Lemma 3.3 and Lemma 3.4]. The proof of Theorem 3 uses some auxiliary results, which concerns with the equation

$$\ddot{u} - \frac{u}{4} + c(e^s)|u(s)|^\beta \operatorname{sgn} u(s) = 0, \ \ s \in [0,\infty), \tag{8}$$

where "$\cdot$" denotes the derivative with respect to the variable s.

Lemma 1. *The change of variable*

$$x(t) = t^{1/2} u(s), \quad s = \log t, \ \ t \in [1,\infty), \tag{9}$$

transforms equation (2) *into equation* (8). *Moreover, equation* (8) *has two types of nonoscillatory solutions. Namely:*

Type (1): *solution u satisfies for large s*

$$0 < |u(s)| \le De^{-s/2} \tag{10}$$

where $|u|$ is decreasing and $D > 0$ is a suitable constant.

Type (2): *solution u intersects the function*

$$Z(s) = (4c(e^s))^{\frac{1}{1-\beta}} \tag{11}$$

infinitely many times, i.e., there exists a sequence $\{s_n\}_{n=1}^{\infty}$, $\lim\limits_n s_n = \infty$ such that

$$|u(s_n)| = Z(s_n).$$

Observe that solutions u of Type (1) in Lemma 1 correspond, via the transformation (9), to subdominant solutions of equation (2) because

$$x(t) = t^{1/2}u(s) \le t^{1/2}K_2e^{-s/2} = K_2,$$

while solutions u of Type (2) correspond to intermediate solutions of (2).

Concluding remark. Consider equation (2) with

$$b(t) = t^{-(\beta+3)/2}t^{\lambda}, \;\; 0 < \lambda < \frac{\beta-1}{2}\,.$$

Then Theorem 3 is not applicable. However, it is possible to construct equation for which intermediate solutions exist. Observe that for $\lambda = (\beta-1)/2$ all solutions of such equation are oscillatory. How to relax conditions (6) in order to exist intermediate solutions?

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