Description of the Perron Exponent of a Linear Differential System with Unbounded Coefficients as a Function of the Initial Vector of a Solution

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1 Introduction

For a given $n \in \mathbb{N}$, let $\widetilde{\mathcal{M}}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with piecewise continuous coefficients, which are matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$, and by $\mathcal{M}_n$ its subclass that consists of systems with coefficients that are bounded on the semiaxis $\mathbb{R}_+$. In what follows, we identify system (1.1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$ and the like. The vector space of solutions of system (1.1) will be denoted by $\mathcal{S}(A)$ and the set of its nonzero solutions by $\mathcal{S}_*(A)$ (i.e. $\mathcal{S}_*(A) = \mathcal{S}(A) \setminus \{\mathbf{0}\}$).

The following definition is due to O. Perron [19].

The lower exponent of a vector-function $x : \mathbb{R}_+ \to \mathbb{R}^n_* \equiv \mathbb{R}^n \setminus \{\mathbf{0}\}$ is the quantity

$$\pi[x] = \varliminf_{t\to+\infty} \frac{1}{t} \ln \|x(t)\|. \tag{1.2}$$

Note that a lower exponent may well be infinite and hence is generally a point of the extended real line $\overline{\mathbb{R}} \equiv \mathbb{R} \sqcup \{-\infty, +\infty\}$, which we equip with the standard order and the order topology.

Assigning to each solution $x(\,\cdot\,) \in \mathcal{S}_*(A)$ of system (1.1) its lower exponent $\pi[x]$, we obtain the functional $\pi^A : \mathcal{S}_*(A) \to \overline{\mathbb{R}}$, which is called *the lower exponent* of system (1.1). Obviously, if $A \in \mathcal{M}_n$, then the range of the functional π^A is contained in a bounded interval. Such defined functionals π^A have different domains, which is not always convenient. In order to have a unified view of these functionals and to make it possible to compare them with one another, they are put in one-to-one correspondence with functions defined on $\mathbb{R}^n_*$. Namely, there is a natural isomorphism $\iota_A : \mathbb{R}^n \to \mathcal{S}(A)$ defined by $\xi \mapsto x(\,\cdot\,, \xi)$, where $x(\,\cdot\,, \xi)$ is the solution of the system A starting at the initial moment $t = 0$ from the vector $\xi \in \mathbb{R}^n$. Then the function $\pi_A : \mathbb{R}^n_* \to \overline{\mathbb{R}}$ defined by $\pi_A = \pi^A \circ i_A$ is called *the Perron exponent* of system (1.1). As said above, π_A takes finite values and is bounded wherever $A \in \mathcal{M}_n$.

The Perron exponent is one of a number of asymptotic characteristics, which are functionals defined on solutions of differential systems and reflecting one or another of their qualitative or asymptotic properties. Historically, the first in this series was the Lyapunov exponent λ_A [18], which is of fundamental importance in the stability theory. As is well known, it is defined similarly to the Perron exponent with the replacement in (1.2) of the lower limit by the upper one. Several different asymptotic characteristics are proposed by I. N. Sergeev (see, e.g., [20, 21]).

Lower exponents are introduced by O. Perron [19] by analogy with Lyapunov characteristic exponents, and he was the first to observe that some of their properties differ from those of Lyapunov exponents. At the same time no serious research of qualitative properties of solutions that Perron exponents represent has been carried out until recently. Situation has changed since the publication of works [22, 23], in which Perron stability was defined and some properties of this notion were treated.

The first problem that arises when studying an asymptotic characteristic is to completely describe it as a function of an initial vector for linear differential systems, that is, for example, for the Perron exponent it is required to obtain a complete description of the following function classes:

$$\mathcal{P}_n = \big\{\pi_A: \ A \in \mathcal{M}_n\big\} \ \text{and} \ \widetilde{\mathcal{P}}_n = \big\{\pi_A: \ A \in \widetilde{\mathcal{M}}_n\big\}. \tag{1.3}$$

To date solutions to these problems are only known for the Lyapunov exponent [18], [7, p. 25–26] and the lower and upper Bohl exponents [5, 6] (definition of the two latter ones see in [8, p. 171–172], [24]). It is worth remarking that whereas a description of the Lyapunov exponent involves linear algebra concepts, that of the Bohl exponents requires the language of descriptive function theory, and so is the case for classes (1.3). A description of the second of these is given in the report, while a description of the first one is unknown so far.

Let us provide a number of properties of the Perron exponent that demonstrate its fundamental difference from the Lyapunov exponent, which seems outwardly similar.

A. M. Lyapunov [18], [7, p. 30] established that the number of different values that the Lyapunov exponent of a system $A \in \mathcal{M}_n$ takes does not exceed its dimension n. O. Perron discovered [19] that for his eponymous exponent this is not the case. He gave an example of a two-dimensional diagonal system with bounded coefficients whose Perron exponent takes exactly three different values. N. A. Izobov showed [15] that the Perron exponent of a diagonal system (1.1) takes no more than $2^n - 1$ values, and in the work [1] for every integer $m \in [1, 2^n - 1]$ a diagonal system (1.1) is constructed such that its Perron exponent takes exactly m different values.

For non-diagonal systems the structure of the range of the Perron exponent may be much more complicated: in the work [16] a system is constructed such that the lower exponents of its solutions fill an entire interval, and in [2] it is proved that a set P is the range of the Perron exponent of a system $A \in \mathcal{M}_n$ if and only if P is a bounded Suslin set containing its sup.

Despite these differences in structure of the ranges of the Lyapunov and Perron exponents of system (1.1), Lebesgue sets of restrictions of these functionals to affine subspaces have some similarity. So, N. A. Izobov established [15, 17] that for any $A \in \mathcal{M}_n$ and affine subspace $\Pi_k \subset \mathbb{R}^n$ of dimension k $(1 \le k \le n)$ the set

$$P(\Pi_k) \equiv \Big\{\xi \in \Pi_k \setminus \{\mathbf{0}\}: \ \ \pi(\xi) < \sup_{\zeta \in \Pi_k \setminus \{\mathbf{0}\}} \pi(\zeta)\Big\}$$

has zero k-dimensional Lebesgue measure, i.e.

$$\operatorname{mes} P(\Pi_k) = 0. \tag{1.4}$$

In other words, the set of lower exponents of solutions with the initial vectors from an affine plane Π_k contains its sup, and for almost all (with respect to Lebesgue measure) initial vectors from Π_k, the corresponding solutions starting from them have lower exponents equal to this sup. For one-dimensional affine subspaces the specified property can be strengthened [3]: for any affine line Π_1, the set $P(\Pi_1)$ has zero Hausdorf $\ln^{\nu} |\ln(\,\cdot\,)|$-measure for all $\nu < -1$.

It is easy to verify [4] that $P(\Pi_1)$ is a $G_{\delta\sigma}$-set and π_A is Baire 2. These statements and the above-cited statements from [3] are unimprovable [4]: for each $n \geq 2$ there exists a system $A \in \mathcal{M}_n$ such that for a certain line Π_1 the set $P(\Pi_1)$ is exactly $G_{\delta\sigma}$ with infinite Hausdorf $\ln^{-1}|\ln(\,\cdot\,)|$-measure and the function π_A is exactly Baire 2.

A. G. Gargyants [10] discovered that for systems in $\widetilde{\mathcal{M}}_n \setminus \mathcal{M}_n$ for $n \geqslant 2$ property (1.4) is generally not valid: he constructed a system $A \in \widetilde{\mathcal{M}}^n$ such that for any k-dimensional $(1 \leq k \leq n)$ affine subspace $\Pi_k \subset \mathbb{R}^n$, different from a line containing the origin, the set $\Pi_k \setminus P(\Pi_k)$ has zero k-dimensional Lebesgue measure and is of the first Baire category with respect to Π_k. He also proved [12] that property (1.4) holds for all systems $A \in \widetilde{\mathcal{M}}^n$ satisfying

$$\overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|A(t)\| \leq 0.$$

A natural question arises: is the set $P(\Pi_k)$ of the first Baire category with respect to Π_k for any $n \geqslant 2$, $k = 1, \dots, n$, and system in $\mathcal{M}_n$? The answer (in the negative) was obtained by A. G. Gargyants [14]: for each $n \geq 2$ there exists a system $A \in \mathcal{M}_n$ such that the set $\mathbb{R}^n \setminus P(\mathbb{R}^n)$ is of the first Baire category.

2 The main result

The problem is to obtain for each $n \geq 2$ a set-theoretic description of Perron exponents of systems in $\widetilde{\mathcal{M}}_n$, i.e. of the function class $\widetilde{\mathcal{P}}_n$ defined in (1.3). Note that a description of the classes $\mathcal{P}_1$ and $\widetilde{\mathcal{P}}_1$ is trivial: they consist of all constant functions $\mathbb{R}^1_* \to \mathbb{R}$ and $\mathbb{R}^1_* \to \overline{\mathbb{R}}$, respectively.

A. G. Gargyants obtained [11, 13] progress in this problem: he proved that for any $n \geq 2$ the class $\widetilde{\mathcal{P}}_n$contains all continuous functions $f : \mathbb{R}^n_* \to \mathbb{R}$ satisfying the condition

$$f(c\xi) = f(\xi), \quad \xi \in \mathbb{R}^n_*, \quad c \in \mathbb{R}_*. \tag{2.1}$$

In [9], this result was extended to upper semicontinuous functions.

For every $n \geq 2$, a complete description of the class $\widetilde{\mathcal{P}}_n$ is provided by the following

Theorem. *A function $f : \mathbb{R}^n_* \to \overline{\mathbb{R}}$ belongs to the class $\widetilde{\mathcal{P}}_n$ if and only if it satisfies* (2.1) *and for all $r \in \mathbb{R}$, the inverse image $f^{-1}([-\infty, r])$ of the closed ray $[-\infty, r]$ is a G_δ-set in $\mathbb{R}^n_*$.*

References

[1] E. A. Barabanov, Attainability of the bound for the number of lower exponents of a linear differential diagonal system. (Russian) *Dokl. Akad. Nauk BSSR* **26** (1982), no. 12, 1069–1072.

[2] E. A. Barabanov, The structure of the set of lower Perron exponents of a linear differential system. (Russian) *Differentsial'nye Uravneniya* **22** (1986), no. 11, 1843–1853.

[3] E. A. Barabanov, Distribution of lower Perron exponents of linear differential systems on lines in a phase space. (Russian) *Differentsial'nye Uravneniya* **24** (1988), no. 12, 2042–2046; translation in *Differential Equations* **24** (1988), no. 12, 1367–1371 (1989).

[4] E. A. Barabanov, Sharpness of some statements on lower Perron exponents. (Russian) *Dokl. Akad. Nauk BSSR* **34** (1990), no. 3, 200–203.

[5] E. A. Barabanov and A. V. Konyukh, Uniform exponents of linear systems of differential equations. (Russian) *Differentsial'nye Uravneniya* **30** (1994), no. 10, 1665–1676; translation in *Differential Equations* **30** (1994), no. 10, 1536–1545 (1995).

[6] E. A. Barabanov and A. V. Konyukh, Bohl exponents of linear differential systems. *Mem. Differential Equations Math. Phys.* **24** (2001), 151–158.

[7] B. F. Bylov, R. È. Vinograd, D. M. Grobman and V. V. Nemyckiĭ, *Theory of Ljapunov Exponents and its Application to Problems of Stability.* (Russian) Izdat. "Nauka", Moscow, 1966.

[8] Ju. L. Dalec'kiĭ and M. G. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space.* Translations of Mathematical Monographs, Vol. 43. American Mathematical Society, Providence, R.I., 1974.

[9] E. I. Fominykh and A. F. Kasabutsky, On distribution of the values of the Perron exponent of solutions to linear differential systems with unbounded coefficients. (Russian) *XVIII Intl. Sci. Conf. on Differ. Equ. (Erugin's readings – 2018): Abstracts of Intl. Sci. Conf. Grodno, May 15-18, 2018.* **83** (2018); *Inst. of Math., NAS of Belarus, Minsk*, 2018, 58–59.

[10] A. G. Gargyants, A remark on the problem of the typicality and essentiality of values of the Perron exponent of unbounded linear systems. (Russian) *Differ. Uravn.* **49** (2013), No. 11, 1505–1506.

[11] A. G. Gargyants, On the distribution of values of Perron exponents over solutions to systems with unbounded coefficients. (Russian) *Differ. Uravn.* **53** (2017), no. 11, 1567.

[12] A. G. Gargyants, Supremum of the Perron exponent on the solutions of a linear system with slowly growing coefficients is metrically typical. (Russian) *Differ. Uravn.* **54** (2018), no. 8, 1011–1017; translation in *Differ. Equ.* **54** (2018), no. 8, 993–999.

[13] A. G. Gargyants, On the existence of a linear differential system with given Perron exponents. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **83** (2019), no. 2, 21–39; translation in *Izv. Math.* **83** (2019), no. 2, 214–231.

[14] A. G. Gargyants, On the lack of topologically essential values of the Perron exponent on solutions to a linear differential system with bounded coefficients. (Russian) *Differ. Uravn.* **56** (2020), no. 1, 53–61; translation in *Differ. Equ.* **56** (2020), no. 1, 51–59.

[15] N. A. Izobov, On a set of lower indices of a linear differential system. (Russian) *Differencial'nye Uravnenija* **1** (1965), 469–477.

[16] N. A. Izobov, The set of lower exponents of positive measure. (Russian) *Differencial'nye Uravnenija* **4** (1968), 1147–1149.

[17] N. A. Izobov, The measure of the solution set of a linear system with the largest lower exponent. (Russian) *Differentsial'nye Uravneniya* **24** (1988), no. 12, 2168–2170.

[18] A. M. Lyapunov, The general problem of the stability of motion. *Int. J. Control* **55** (1992), no. 3, 531–773.

[19] O. Perron, Die ordnungszahlen linearer differentialgleichungssysteme. (German) *Math. Z.* **31** (1930), no. 1, 748–766.

[20] I. N. Sergeev, Properties of characteristic frequencies of linear equations of arbitrary order. (Russian) *Tr. Semin. im. I. G. Petrovskogo* No. 29 (2013), Part II, 414- 442; translation in *J. Math. Sci. (N.Y.)* **197** (2014), no. 3, 410–426.

[21] I. N. Sergeev, Lyapunov characteristics of oscillation, rotation, and wandering of solutions of differential systems. (Russian) *Tr. Semin. im. I. G. Petrovskogo* No. 31 (2016) 177 -219; translation in *J. Math. Sci. (N.Y.)* **234** (2018), no. 4, 497–522.

[22] I. N. Sergeev, Perron stability and its study at the first approximation. (Russian) *Dokl. Akad. Nauk, Ross. Akad. Nauk* **486** (2019), no. 1, 20–23; translation in *Dokl. Math.* **99** (2019), no. 3, 252–254.

[23] I. N. Sergeev, Definition and some properties of Perron stability. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 636 -646; translation in *Differ. Equ.* **55** (2019), no. 5, 620–630.

[24] R. E. Vinograd, Simultaneous attainability of central Lyapunov and Bohl exponents for ODE linear systems. *Proc. Amer. Math. Soc.* **88** (1983), no. 4, 595–601.