Remark on Continuous Dependence of Solutions to the Riccati equation on its Righthand Side

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Abstract. The Riccati equation is considered to show some special features of continuous dependence on the right-hand side of the equation.

Consider the initial value problem for the Riccati equation

$$u' + u^2 = K(t), \quad K \in C[0;T], \tag{1}$$

with the initial condition $u(0) = u_0$. Suppose that the problem has a solution on $[0;T]$. We are interested whether the existence of solutions on $[0;T]$ still holds under small perturbation of the right-hand side. The formulation is not strict enough and admits opposite answers for its different treatments.

We have the following result detailing, for the case under consideration, the classical theorem on continuous dependence of solutions on the right-hand side and initial conditions (see, for example, [1, Chapter 7, Theorem 6]).

Theorem 1. *Let $u(t)$ defined on $[0;T]$ be a solution to equation* (1) *with $u(0) = u_0$. Then for each function $F(t) \in C[0,T]$, satisfying on $[0;T]$ the condition*

$$|K(t) - F(t)| < \varepsilon = \left(4 \int\limits_0^T \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) dt \cdot \int\limits_0^T \exp\left(2 \int\limits_0^t u(\tau)\,d\tau\right) dt\right)^{-1},$$

the initial value problem

$$v' + v^2 = F(t), \quad v(0) = u_0 \tag{2}$$

also has a solution defined on $[0;T]$.

Proof. We are looking for the solution v having the form

$$v(t) = u(t) + z(t) \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) \tag{3}$$

with $z(t)$ to be determined. Immediate calculations show that

$$v' = u' + z' \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right) - 2zu \exp\left(-2 \int\limits_0^t u(\tau)\,d\tau\right)$$

$$= K - u^2 + z' \exp\bigg(-2\int\limits_0^t u(\tau)\,d\tau\bigg) - 2zu\exp\bigg(-2\int\limits_0^t u(\tau)\,d\tau\bigg)$$

$$= F - v^2 = F - u^2 - 2zu\exp\bigg(-2\int\limits_0^t u(\tau)\,d\tau\bigg) - z^2\exp\bigg(-4\int\limits_0^t u(\tau)\,d\tau\bigg),$$

whence

$$z' + z^2\exp\bigg(-2\int\limits_0^t u(\tau)\,d\tau\bigg) + (K-F)\exp\bigg(2\int\limits_0^t u(\tau)\,d\tau\bigg) = 0. \tag{4}$$

Hereafter we use the following notation:

$$2\int\limits_0^t u(\tau)\,d\tau = U(t), \quad \exp U(t) = E(t).$$

So, equation (4) can be rewritten as

$$z' + z^2 E^{-1} + (F-K)E = 0.$$

To find such z, we use a contracting operator Φ acting on the space Z_δ of all continuous functions z satisfying $|z(t)| \le \delta$ on $[0;T]$ with some $\delta > 0$. We define Φ by

$$\Phi(z)(t) = \int\limits_0^t \Big(E(\tau)(K(\tau) - F(\tau)) - z^2(\tau)E(\tau)^{-1}\Big)\,d\tau$$

and have to check that

(i) $\Phi(z) \in Z_\delta$ whenever $z \in Z_\delta$

and

(ii) Φ is contracting.

First we prove (ii).

$$\big|\Phi(z_1)(t) - \Phi(z_2(t)\big| = \int\limits_0^t \big|E(\tau)^{-1}\big|\,\big|z_2^2(\tau) - z_1^2(\tau)\big|\,d\tau$$

$$= \int\limits_0^t E(\tau)^{-1}\big|z_2(\tau) - z_1(\tau)\big|\,\big|z_2(\tau) + z_1(\tau)\big|\,d\tau \le 2\delta\int\limits_0^t E(\tau)^{-1}\big|z_2(\tau) - z_1(\tau)\big|\,d\tau$$

$$\le 2\delta\int\limits_0^t E(\tau)^{-1}\sup_{\tau\in[0,T]}\big|z_2(\tau) - z_1(\tau)\big|\,d\tau = 2\delta\int\limits_0^t E(\tau)^{-1}\,d\tau\cdot\|z_1 - z_2\|.$$

The operator Φ is contracting if $\delta\int\limits_0^t E(\tau)^{-1}\,d\tau < 1$ on $[0;T]$, i.e.

$$\delta < \frac{1}{2}\bigg(\int\limits_0^T E(\tau)^{-1}\,d\tau\bigg)^{-1}.$$

Now we take such δ and prove (i). Suppose $|K(\tau)-E(\tau)|<\varepsilon$ and $\|z\|\le\delta$. Then

$$|\Phi(z)(t)|\le\varepsilon\int\limits_0^T E(\tau)\,d\tau+\delta^2\int\limits_0^T E(\tau)^{-1}\,d\tau<\varepsilon\int\limits_0^T E(\tau)\,d\tau+\frac{\delta}{2}\,,$$

which is less or equal to δ if

$$\varepsilon\le\frac{\delta}{2}\,\bigg(\int\limits_0^T E(\tau)\,d\tau\bigg)^{-1}.$$

So, if we have a solution u to equation (1) with $u(0)=u_0$, then we can obtain E and $\int\limits_0^T E(\tau)\,d\tau$. Now we have to find δ such that

$$2\varepsilon\int\limits_0^T E(\tau)\,d\tau\le\delta<\frac{1}{2}\,\bigg(\int\limits_0^T E(\tau)^{-1}\,d\tau\bigg)^{-1}.$$

This is possible whenever

$$\varepsilon<\frac{1}{4}\bigg(\int\limits_0^T E(\tau)^{-1}\,d\tau\bigg)^{-1}\bigg(\int\limits_0^T E(\tau)\,d\tau\bigg)^{-1}.$$

If the estimates are valid, then there exists $z\in Z_\delta$ such that $\Phi(z)=z$, whence $z'=E(K-F)-z^2E^{-1}$ and $z(0)=0$, i.e. the function v defined by (3) is a solution to the related Riccati equation and $v(0)=u(0)$. $\square$

Note that the existence of a solution to (2) in the proof depends not only on the difference of the functions K and F but also on the solution u itself. This is not just a defect of the proof as shown in the following result.

Theorem 2. *If $T=\pi/2$ and $K(t)\equiv-1$ in* (1)*, then for each $\varepsilon>0$ there exist an initial value u_0 and a continuous function F on $[0;T]$ such that $\|F-K\|<\varepsilon$, and equation* (1) *has a solution $u\in C^1[0;T]$ with $u(0)=u_0$, while there is no solution to* (2) *on $[0;T]$.*

Proof. First we can solve the equation $u'+u^2=-A^2$ with arbitrary $A\neq0$ to obtain the solution $u=A\tan(At_0-At)$.

In the case $A=1$ we have a solution $u(t)=\tan(t_0-t)$ to equation (1) defined in particular for all t satisfying $-T<t_0-t<T$. If $0<t_0<T$, then the solution u is defined, inter alia, on the segment $[0;T]$.

Now consider $A=1+\varepsilon$ and $F(t)=-(1+\varepsilon)^2$. The function

$$v=(1+\varepsilon)\,\tan\big((1+\varepsilon)t_1-(1+\varepsilon)t\big)$$

is a solution to (2) provided that

$$v(0)=u(0)=\tan t_0=(1+\varepsilon)\tan((1+\varepsilon)t_1).$$

Thus,

$$t_1=\frac{1}{1+\varepsilon}\,\arctan\Big(\frac{\tan t_0}{1+\varepsilon}\Big).$$

The solution is bounded on $[0;T]$ if $(1+\varepsilon)t_1-(1+\varepsilon)t\in(-T;T)$ whenever $t\in[0;T]$.

For $t=0$ we have $(1+\varepsilon)t_1\in(-T;T)$.

For $t=T$ we need

$$-T<(1+\varepsilon)t_1-(1+\varepsilon)T<T,$$

whence

$$\varepsilon T<(1+\varepsilon)t_1=\arctan\Big(\frac{\tan t_0}{1+\varepsilon}\Big),$$

and therefore $\tan t_0>(1+\varepsilon)\,\tan\varepsilon T$.

So, if the constant t_0 does not satisfy this condition, then v is not defined on the whole segment $[0;T]$. For arbitrary small $\varepsilon>0$ there exists sufficiently small $t_0>0$ making the last inequality false. $\square$

So, no estimate based just on the difference $K(t)-F(t)$ is possible to provide the existence of a solution to problem (2) for all u_0.

Note also that Theorem 1 becomes wrong if we replace $[0;T]$ with $[0;T)$.

Acknowledgement

This work was partially supported by RSF (Project # 20-11-20272.)

References

[1] A. F. Filippov, *Introduction to the Theory of Differential Equations.* (Russian) URSS, 2004.