A Problem for a Family of Partial Integro-Differential Equations with Weakly Singular Kernels

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On the domain $\Omega=[0,T]\times[0,\omega]$, we consider the family of problems for the system of partial integro-differential equations with weakly singular kernels

$$\frac{\partial v}{\partial t}=A(t,x)v+\int\limits_0^T K(t,s,x)v(s,x)\,ds+f(t,x), \tag{1}$$

$$P(x)v(0,x)+S(x)v(T,x)=\varphi(x),\ \ x\in[0,\omega], \tag{2}$$

where $v(t,x)=col(v_1(t,x),v_2(t,x),\dots,v_n(t,x))$ is an unknown vector function, the $(n\times n)$ matrix $A(t,x)$, and n vector function $f(t,x)$ are continuous on Ω, the $(n\times n)$ matrix $K(t,s,x)$ has the form $K(t,s,x)=\frac{1}{|t-s|^\alpha}\,H(t,s,x)$, and the $(n\times n)$ matrix $H(t,s,x)$ is continuous on $[0,T]\times[0,T]\times[0,\omega]$, $0<\alpha<1$, the $(n\times n)$ matrices $P(x)$, $S(x)$ and n vector function $\varphi(x)$ are continuous on $[0,\omega]$.

A continuous function $v:\Omega\to\mathbb{R}^n$ that has a continuous derivative with respect to t on Ω is called a solution to the family problems for the system of integro-differential equations (1), (2) if it satisfies system (1) and condition (2) for all $(t,x)\in\Omega$ and $x\in[0,\omega]$, respectively.

Partial integro-differential equations and various problems for them are arisen as mathematical models of various physical processes [2, 3, 25]. Boundary value problems for ordinary integro-differential equations with continuous kernels and weakly singular or other nonsmooth kernels were researched in [1, 4–11, 13, 14, 16–24] by the different methods. Some problems for partial integro-differential equations with singular kernels were considered in [26–33].

Nevertheless, the establishment of conditions for the solvability of the family problems for partial integro-differential equations with weakly singular kernels is an actual problem.

The aim of the present communication is to apply the Dzhumabaev parametrization method [12] and the results of article [4] to the family of partial integro-differential equations with weakly singular kernels.

For this we construct a homogeneous family of partial integral equations with weakly singular kernels of the second kind and introduce an analog of regular partition for Ω by the initial data of the partial integro-differential equation (1).

For fixed $x\in[0,\omega]$ problem (1), (2) is a linear problem for the system of integro-differential equations with weakly kernels. Suppose a variable x is changed on $[0,\omega]$; then we obtain a family of problems for partial integro-differential equations with weakly kernels.

Let us divide domain Ω equally into N parts and denote this partition by Δ_N:

$$\Delta_N=\big\{t_0=0<t_1<\dots<t_N=T,\ 0\le x\le\omega\big\},$$

where $t_s=sT/N$.

By $v_r(t,x)$ we denote the restriction of the function $v(t,x)$ to the r-th domain $\Omega_r=[t_{r-1},t_r)\times[0,\omega]$, i.e. $v_r(t,x)=v(t,x)$, $(t,x)\in\Omega_r$, $r=1:N$.

Inputting the functional parameters $\lambda_r(x)\widehat{=}v_r(t_{r-1},x)$ and performing the substitution of functions $z_r(t,x)=v_r(t,x)-\lambda_r(x)$ in each of the r-th domain, we obtain the following problem with parameters:

$$\frac{\partial z_r}{\partial t}=A(t,x)(z_r+\lambda_r(x))+\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(t,s,x)(z_j(s,x)+\lambda_j(x))\,ds+f(t,x),\ \ (t,x)\in\Omega_r, \tag{3}$$

$$z_r(t_{r-1},x)=0,\ \ x\in[0,\omega],\ \ r=1:N, \tag{4}$$

$$P(x)\lambda_1(x)+S(x)\lambda_N(x)+S(x)\lim_{t\to T-0}z_N(t,x)=\varphi(x),\ \ x\in[0,\omega], \tag{5}$$

$$\lambda_p(x)+\lim_{t\to t_p-0}z_p(t,x)-\lambda_{p+1}(x)=0,\ \ x\in[0,\omega],\ \ p=1:(N-1), \tag{6}$$

where (6) are the conditions of continuity for the solution at the inner lines of the partition Δ_N.

Introduction of additional parameters [4–11] lets us obtain the initial data (5). Thus, it is possible to determine the system of functions $z([t],x)$ from the family of special Cauchy problems for the systems of integro-differential equations with weakly singular kernels (5), (6) for fixed values of the parameters $\lambda(x)\in C([0,\omega],\mathbb{R}^{nN})$. By using the fundamental matrix $U(t,x)$ of the differential equation $\frac{\partial v}{\partial t}=A(t,x)v$, we reduce problem (5), (6) to the equivalent system of integral equations

$$\begin{aligned}z_r(t,x)&=U(t,x)\int\limits_{t_{r-1}}^{t}U^{-1}(\tau_1,x)\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(\tau_1,s,x)(z_j(s,x)+\lambda_j(x))\,ds\,d\tau_1\\&\quad+U(t,x)\int\limits_{t_{r-1}}^{t}U^{-1}(\tau_1,x)\big[A(\tau_1,x)\lambda_r(x)+f(\tau_1,x)\big]\,d\tau_1,\ \ (t,x)\in\Omega_r,\ \ r=1:N.\end{aligned} \tag{7}$$

Introduce the notations

$$\Phi(\Delta_N,t,x,\alpha)=\sum_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j}K(t,s,x)z_j(s,x)\,ds,$$

$$M(\Delta_N,t,x,\tau,\alpha)=\int\limits_{\tau}^{t_j}K(t,\tau_1,x)U(\tau_1,x)\,d\tau_1U^{-1}(\tau,x),\quad (t,x)\in\Omega,\ \ \tau\in[t_{j-1},t_j),\ \ j=1:N,$$

$$M(\Delta_N,t,x,T,\alpha)=0.$$

Consider the following family of integral equations of the second kind with weakly singular kernel

$$\Phi(\Delta_N,t,x,\alpha)=\int\limits_0^T M(\Delta_N,t,x,\tau,\alpha)\Phi(\Delta_N,\tau,x,\alpha)\,d\tau+D(\Delta_N,t,x,\alpha)\lambda+F(\Delta_N,t,x,\alpha), \tag{8}$$

and the corresponding homogeneous family of integral equations

$$\Phi(\Delta_N,t,x,\alpha)=\int\limits_0^T M(\Delta_N,t,x,\tau,\alpha)\Phi(\Delta_N,\tau,x,\alpha)\,d\tau,\ \ (t,x)\in\Omega. \tag{9}$$

Definition. A partition Δ_N is called regular if the family of integral equations with weakly singular kernel (9) has only a trivial solution.

The set of regular partitions Δ_N is denoted by $\sigma([0,T],x,\alpha)$. As it is known from the theory of integral equations with singular kernels [15], if $\Delta_N \in \sigma([0,T],x,\alpha)$, then (8) has a unique solution for any $\lambda(x) \in C([0,\omega],\mathbb{R}^{nN})$, $F(\Delta_N,t,x,\alpha) \in C(\Omega,\mathbb{R}^n)$, and this solution can be presented in the form

$$\begin{aligned}\Phi(\Delta_N,t,x,\alpha) &= D(\Delta_N,t,x,\alpha)\lambda + F(\Delta_N,t,x,\alpha)\\ &\quad + \int_0^T \Gamma(\Delta_N,t,x,s,1)\big(D(\Delta_N,s,x,\alpha)\lambda(x) + F(\Delta_N,s,x,\alpha)\big)\,ds, \quad (t,x)\in\Omega, \quad (10)\end{aligned}$$

where $\Gamma(\Delta_N,t,x,s,1)$ is the resolvent of the family of integral equations with weakly singular kernel (8). The $n \times nN$ matrix $D(\Delta_N,t,x,\alpha) = (D_r(\Delta_N,t,x,\alpha))$, $r = 1:N$, continuous on Ω, and vector $F(\Delta_N,t,x,\alpha)$ are constructed by the integral representation (7):

$$\begin{aligned}D_r(\Delta_N,t,x,\alpha) &= \int_{t_{r-1}}^{t_r} K(t,\tau,x)U(\tau,x)\int_{t_{r-1}}^{\tau} U^{-1}(\tau_1,x)A(\tau_1,x)\,d\tau_1\,d\tau\\ &\quad + \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} K(t,\tau,x)U(\tau,x)\int_{t_{j-1}}^{\tau} U^{-1}(\tau_1,x)\int_{t_{r-1}}^{t_r} K(\tau_1,s,x)\,ds\,d\tau_1\,d\tau,\\ F(\Delta_N,t,x,\alpha) &= \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} K(t,\tau,x)U(\tau,x)\int_{t_{j-1}}^{\tau} U^{-1}(\tau_1,x)f(\tau_1,x)\,d\tau_1\,d\tau.\end{aligned}$$

Substituting $\sum\limits_{j=1}^{N}\int\limits_{t_{j-1}}^{t_j} K(t,s,x)z_j(s,x)\,ds$ in (7) with the right-hand side of (10), we get the representation of the function $z_r(t,x)$ in terms of $\lambda(x) \in C([0,\omega],\mathbb{R}^{nN})$, $f(t,x) \in C(\Omega,\mathbb{R}^n)$. Then, using this representation, we determine $\lim\limits_{t\to T-0} z_N(t,x)$, $\lim\limits_{t\to t_p-0} z_p(t,x)$, $p = 1:(N-1)$. Substituting these expressions in (5), (6) and multiplying both sides of (5) by $h = \frac{T}{N}$, we get the following linear system of equations for the introduced parameters $\lambda_r(x)$, $r = 1:N$:

$$Q^*(\Delta_N,x,\alpha)\lambda(x) = -F^*(\Delta_N,x,\alpha), \quad \lambda(x) \in C([0,\omega],\mathbb{R}^{nN}). \quad (11)$$

Theorem 1. *If the matrix $Q^*(\Delta_N,x,\alpha) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ in the partition $\Delta_N \in \sigma([0,T],x,\alpha)$ is invertible for all $x \in [0,\omega]$, then the family of problems for the system of partial integro-differential equations with weakly singular kernels* (1), (2) *has a unique solution.*

Theorem 2. *For the unique solvability of family of problems for the system of partial integro-differential equations with weakly singular kernels* (1), (2) *it is necessary and sufficient that the matrix $Q^*(\Delta_N,x,\alpha) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ be invertible for any $\Delta_N \in \sigma([0,T],x,\alpha)$ and for all $x \in [0,\omega]$.*

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