On the Well-Posedness Question of the Cauchy Problem with Weight for Systems of Linear Generalized Ordinary Differential Equations and Impulsive Equations with Singularities

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Let $I=[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerated in the point, $t_0\in\,]a,b[$ and $I_{t_0}=[a,b]\setminus\{t_0\}$, $I_{t_0}^-=[a,t_0[\,$, $I_{t_0}^+=\,]t_0,b]$.

Consider the Cauchy problem for the linear system of generalized ordinary differential equations (GODE) with singularities

$$dx=dA(t)\cdot x+df(t)\ \text{ for }\ t\in I_{t_0}, \tag{1}$$

$$\lim_{t\to t_0-}(H^{-1}(t)\,x(t))=0\ \text{ and }\ \lim_{t\to t_0+}(H^{-1}(t)\,x(t))=0, \tag{2}$$

where $A=(a_{ik})_{i,k=1}^n$ is an $n\times n$-matrix valued function and $f=(f_k)_{k=1}^n$ is an n-vector valued function, both of them have a locally bounded variation on $[a,b]\setminus\{t_0\}$; $H=\operatorname{diag}(h_1,\dots,h_n)$ is a continue diagonal matrix function, continuous and having an inverse $H^{-1}(t)$ for $t\in[a,b]\setminus\{t_0\}$.

Along with system (1) consider the perturbed system

$$dy=d\widetilde{A}(t)\cdot y+d\widetilde{f}(t)\ \text{ for }\ t\in I_{t_0} \tag{3}$$

under condition (2), where $\widetilde{A}$, $\widetilde{f}$ are, as above, a matrix- and vector-functions.

We are interested in the question whether the unique solvability of problem $(1),(2)$ guarantees the unique solvability of problem $(3),(2)$ and nearness of its solutions in the definite sense if matrix-functions A and $\widetilde{A}$ and vector-functions f and $\widetilde{f}$ are nearly among themselves.

The same and related problems for singular linear ordinary differential systems have been investigated in [3] (see also the references therein).

The singularity of system (1) consists in the fact that both A and f need not have bounded variations on any interval containing the point t_0.

The solvability of the singular problem $(1),(2)$ is investigated in [2]. To our knowledge, the well-posedness of $(1),(2)$ has not been considered up to now.

The theory of GODE has been introduced by J. Kurzweil [4]. The interest to the theory has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see $[1,2,4]$ and the references therein).

We present sufficient conditions for the so called H-well-posedness of problem $(1),(2)$. We realize the presented results for systems of impulsive differential equations with fixed points of impulses actions.

We use the following notation and definitions.

$\mathbb{N}=\{1,2,\dots\}$, $\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{ik})_{i,k=1}^{n,m}$ with the standard norm $\|X\|$.

$|X|=(|x_{ik}|)_{i,k=1}^{n,m}$,

$[X]_\pm=\frac{1}{2}\,(|X|\pm X)$; $r(X)$ is the spectral radius of $X\in\mathbb{R}^{n\times n}$. $\mathbb{R}^n=\mathbb{R}^{n\times 1}$.

$O_{n\times m}$ is the zero $n\times m$-matrix, 0_n is the zero n-vector. I_n is the identity $n\times n$-matrix.

$\bigvee\limits_a^b(X)$ is the sum of variations on $[a,b]$ of components of matrix-function X.

$\bigvee\limits_a^{b-}(X)=\lim\limits_{t\to b-}\bigvee\limits_a^t(X)$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point t.

$d_1X(t)=X(t)-X(t-)$, $d_2X(t)=X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all bounded variation matrix-functions.

$\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$ is the set of all $X:I\to\mathbb{R}^{n\times m}$ for which the restriction to $[a,b]$ belong to $\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ for every closed interval $[a,b]$ from I_{t_0}.

If $X(t)=(x_{ik}(t))_{i,k=1}^{n,m}$, then $V(X)(t)=(\bigvee\limits_{a_j}^t(x_{ik}))_{i,k=1}^{n,m}$ for $(t-t_0)(a_j-t_0)>0$ $(j=1,2)$, where $a_1=a$, $a_2=b$.

$[X(t)]_\pm^v\equiv\frac{1}{2}\,(V(X)(t)\pm X(t))$.

$L_{loc}(I_{t_0};\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:I_{t_0}\to\mathbb{R}^{n\times m}$ whose restrictions to every closed interval $[a,b]$ from I_{t_0} is integrable.

s_1,s_2 and s_c are the operators defined, respectively, by

$$s_1(x)(a_j)=s_2(x)(a_j)=0,\;\; s_c(x)(a_j)=x(a_j)\;\;(j=1,2),$$
$$s_1(x)(t)=s_1(x)(s)+\sum_{s<\tau\le t}d_1x(\tau),\;\; s_2(x)(t)=s_2(x)(s)+\sum_{s\le\tau<t}d_2x(\tau),$$
$$s_c(x)(t)=s_c(x)(s)+x(t)-x(s)-\sum_{j=1}^2(s_j(x)(t)-s_j(x)(s))$$
$$\text{for }\; a_1\le s<t<t_0\;\text{ or }\;t_0<s<t\le a_2.$$

If $X\in\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$,

$$\det\big(I_n+(-1)^jd_jX(t)\big)\neq0\;\text{ for }\;t\in I_{t_0}\;\;(j=1,2),$$

and $Y\in\mathrm{BV}_{loc}(I_{t_0};\mathbb{R}^{n\times m})$, then

$$\mathcal{A}(X,Y)(a)=O_{n\times m}\;\;\big(\mathcal{A}(X,Y)(b)=O_{n\times m}\big),$$
$$\mathcal{A}(X,Y)(t)-\mathcal{A}(X,Y)(s)=Y(t)-Y(s)+\sum_{s<\tau\le t}d_1X(\tau)\cdot(I_n-d_1X(\tau))^{-1}\,d_1Y(\tau)$$
$$-\sum_{s\le\tau<t}d_2X(\tau)\cdot(I_n+d_2X(\tau))^{-1}\,d_2Y(\tau)\;\text{ if }\;s<t<t_0\;\;(t_0<s<t).$$

If $g:[a,b]\to\mathbb{R}$ has bounded variation and $x:[a,b]\to\mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau)=\int_{]s,t[}x(\tau)\,ds_c(g)(\tau)+\sum_{s<\tau\le t}x(\tau)\,d_1g(\tau)+\sum_{s\le\tau<t}x(\tau)\,d_2g(\tau),$$

where $\int\limits_{]s,t[}x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_c(g)$. So $\int\limits_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral [4].

Let
$$\int\limits_{s}^{t\pm} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_{s}^{t\pm\delta} x(\tau)\,dg(\tau),\quad \int\limits_{s\pm}^{t} x(\tau)\,dg(\tau)=\lim_{\delta\to 0+}\int\limits_{s\pm\delta}^{t} x(\tau).$$

If $G=(g_{ik})_{i,k=1}^{l,n}$ and $X=(x_{kj})_{k,j=1}^{n,m}$ are matrix-functions on $[a,b]$, then

$$\int\limits_{a}^{t} dG(\tau)\,X(\tau)\equiv\bigg(\sum_{k=1}^{n}\int\limits_{a}^{t} x_{kj}(\tau)dg_{ik}(\tau)\bigg)_{i,j=1}^{l,m},\quad S_j(G)=\big(s_j(g_{ik})\big)_{i,k=1}^{l,n}.$$

A vector-function $x:I_{t_0}\to\mathbb{R}^n$ is said to be a solution of system (1) if $x\in\mathrm{BV}([c,d],\mathbb{R}^n)$ for every closed interval $[c,d]$ from I_{t_0} and

$$x(t)=x(s)+\int\limits_{s}^{t} dA(\tau)\cdot x(\tau)+f(t)-f(s)\ \text{ for }\ c\le s<t\le d.$$

We assume that $\det(I_n+(-1)^j d_jA(t))\neq 0$ for $t\in I_{t_0}$ $(j=1,2)$. The inequalities guarantee the unique solvability of the Cauchy problem for the case when $A\in\mathrm{BV}_{loc}(I,\mathbb{R}^{n\times n})$ and $f\in\mathrm{BV}_{loc}(I,\mathbb{R}^n)$ (see, [4]).

Let $A_0\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ is a matrix-function such that

$$\det\big(I_n+(-1)^j d_jA_0(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2). \tag{4}$$

Then a matrix-function $C_0:I_{t_0}\times I_{t_0}\to\mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system $dx=dA_0(t)\cdot x$ if, for every interval $J\subset I$ and $\tau\in J$, the restriction of $C_0(\,\cdot\,,\tau)$ to J is the fundamental matrix of this system satisfying the condition $C_0(\tau,\tau)=I_n$.

Let
$$I_{t_0}^-(\delta)=[t_0-\delta,t_0[\,,\quad I_{t_0}^+(\delta)=\,]t_0,t_0+\delta],\quad I_{t_0}(\delta)=I_{t_0}^-(\delta)\cup I_{t_0}^+(\delta)\ \ (\delta>0).$$

Definition 1. Problem (1), (2) is said to be H-well-posed with respect to the pair of the matrix-functions $(S_1(A_0),S_2(A_0))$ if it has a unique solution x and for every $\varepsilon>0$ there exists $\eta>0$ such that problem (3), (2) has a unique solution y and the estimate

$$\big\|H(t)\,(x(t)-y(t))\big\|<\varepsilon\ \text{ for }\ t\in I \tag{5}$$

holds for every $\widetilde{A}\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^{n\times n})$ and $\widetilde{f}\in\mathrm{BV}_{loc}(I_{t_0},\mathbb{R}^n)$ such that

$$\det\big(I_n+(-1)^j d_j\widetilde{A}(t)\big)\neq 0\ \text{ for }\ t\in I_{t_0}\ \ (j=1,2),$$
$$\bigg\|\int\limits_{t_0\pm}^{t} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}(A_0,\widetilde{A}-A))(s)\cdot H(s)\bigg\|<\eta\ \text{ for }\ t\in I_{t_0}^{\pm},\ \text{ respectively},$$
$$\bigg\|\int\limits_{t_0\pm}^{t} H^{-1}(s)d\,\mathrm{V}(\mathcal{A}(A_0,\widetilde{f}-f))(s)\bigg\|<\eta\ \text{ for }\ t\in I_{t_0}^{\pm},\ \text{ respectively}.$$

We note that the matrix-functions $S_1(A_0)$ and $S_2(A_0)$ are including in the definition of the operators $V(\mathcal{A}(A_0,\widetilde{A}-A))$ and $V(\mathcal{A}(A_0,\widetilde{f}-f))$.

Theorem 1. *Let there exist a matrix-function $A_0 \in \mathrm{BV}_{loc}(I_{t_0}, \mathbb{R}^{n\times n})$ and constant matrices $B_0. B \in \mathbb{R}_+^{n\times n}$ such that the conditions $r(B) < 1$,*

$$|C_0(t,\tau)| \le H(t)\, B_0\, H^{-1}(\tau) \text{ for } t \in I_{t_0}(\delta), \ \ (t-t_0)(\tau-t_0) > 0, \ \ \frac{|\tau - t_0|}{|t-t_0|} \le 1,$$

$$\bigg| \int\limits_{t_0\pm}^{t} |C_0(t,s)|\, d\,\mathrm{V}(\mathcal{A}(A_0, A - A_0))(s)\, H(s) \bigg| \le H(t) B, \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta)$$

hold for some $\delta > 0$. Let, moreover,

$$\lim_{t\to t_0\pm} \bigg\| \int\limits_{t_0\pm}^{t} H^{-1}(t)\, C_0(t,\tau)\, d\mathcal{A}(A_0, f)(\tau) \bigg\| = 0, \ \ \text{respectively.}$$

Then problem $(1),(2)$ is H-well-posed with respect to $(S_1(A_0), S_2(A_0))$.

Theorem 2. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that the conditions $r(B) < 1$,*

$$\big[(-1)^j d_j a_{ii}(t)\big]_+ > -1 \ \text{ for } \ t < t_0, \quad [(-1)^j d_j a_{ii}(t)]_- < 1 \ \text{ for } \ t > t_0 \ \ (j = 1,2),$$

$$c_i(t,\tau) \le b_0 \frac{h_i(t)}{h_i(\tau)} \ \text{ for } \ t \in I_{t_0}(\delta), \quad (t-t_0)(\tau-t_0) > 0, \ \ |\tau-t_0| \le |t-t_0|,$$

$$\bigg| \int\limits_{t_0\pm}^{t} c_i(t,\tau) h_i(\tau)\, d\big[a_{ii}(\tau)\,\mathrm{sgn}(\tau - t_0)\big]_+^v \bigg| \le b_{ii}\, h_i(t), \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta),$$

$$\bigg| \int\limits_{t_0\pm}^{t} c_i(t,\tau) h_k(\tau)\, d\,\mathrm{V}(\mathcal{A}(a_{0ii}, a_{ik}))(\tau) \bigg| \le b_{ik}\, h_i(t), \ \ \text{respectively, on } I_{t_0}^{\pm}(\delta)$$

$(i \ne k;\ i,k = 1,\dots,n)$ hold for some $b_0 > 0$ and $\delta > 0$. Let, moreover,

$$\lim_{t\to t_0\pm} \int\limits_{t_0\pm}^{t} \frac{c_i(t,\tau)}{h_i(t)}\, d\,\mathrm{V}(\mathcal{A}(a_{0ii}, f_i))(\tau) = 0, \ \ \text{respectively} \ \ (i = 1,\dots,n),$$

where

$$a_{0ii}(t) \equiv -\big[a_{ii}(t)\,\mathrm{sgn}(t-t_0)\big]_-^v\, \mathrm{sgn}(t-t_0) \ \ (i = 1,\dots,n),$$

and c_i is the Cauchy function of the equation $dx = x\, da_{0ii}(t)$ for $i \in \{1,\dots,n\}$. Then problem $(1),(2)$ is H-well-posed with respect to the pair $(S_1(A_0), S_2(A_0))$, where

$$A_0(t) \equiv \mathrm{diag}(a_{011}(t),\dots,a_{0nn}(t)).$$

The Cauchy functions $c_i(t,\tau)$ $(i = 1,\dots,n)$, mentioned in the theorem, have the well known form (see, for example, [1]).

Now we apply the previous results for the Cauchy problem with weight for the singular impulsive differential system

$$\frac{dx}{dt} = P(t)x + q(t), \quad x(\tau_l+) - x(\tau_l-) = G(l)x(\tau_l) + g(l) \ \ (l \in \mathbb{N}); \tag{6}$$

$$\lim_{t\to b-} (H^{-1}(t)x(t)) = 0, \tag{7}$$

where $\tau_l \in [a,b[$ $(l \in \mathbb{N})$, $\lim\limits_{l\to+\infty} \tau_l = b$, are points of fixed impulses actions, $P = (p_{ik})_{i,k=1}^n \in L_{loc}([a,b[;\mathbb{R}^{n\times n})$, $q = (q_k)_{k=1}^n \in L_{loc}([a,b[;\mathbb{R}^n)$, and $G = (g_{ik})_{i,k=1}^n \in E(\mathbb{N};\mathbb{R}^{n\times n})$, $g = (g_k)_{k=1}^n \in E(\mathbb{N};\mathbb{R}^n)$.

We assume that $T = \{\tau_1, \tau_2, \dots\}$, $\mathrm{AC}_{loc}([a,b[\setminus T;\mathbb{R}^{n\times m})$ is the matrix-function whose restrictions to every $[c,d] \subset [a,b[\setminus T$ is absolutely continuous. $E(\mathbb{N};\mathbb{R}^{n\times m})$ is the set of all discrete matrix-functions from $\mathbb{N}$ into $\mathbb{R}^{n\times m}$. $N_{\alpha,\beta} = \{l \in \mathbb{N} : \alpha \le \tau_l < \beta\}$.

Let $G_0 \in E(\mathbb{N};\mathbb{R}^{n\times n})$ be such that

$$\det(I_n + G_0(l)) \neq 0 \ \ (l \in \mathbb{N}).$$

Then for every $X \in L_{loc}([a,b[;\mathbb{R}^{n\times n})$ and $Y \in E(\mathbb{N};\mathbb{R}^{n\times n})$ we put

$$\mathcal{A}_\iota(G_0;X,Y)(t) \equiv \int\limits_a^t X(\tau)\,d\tau + \sum_{l\in N_{a,t}} (I_n + G_0(l))^{-1} Y(\tau_l).$$

A vector-function $x \in \mathrm{AC}_{loc}([a,b[\setminus T;\mathbb{R}^n)$ is said to be a solution of system (6) if $x'(t) = P(t)x(t) + q(t)$ for a.a. $t \in [a,b[\setminus T$ and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,2,\dots)$ satisfying (8). In addition, x is a solution of system (6) if and only if it is a solution of (1), where

$$A(t) \equiv \int\limits_a^t P(\tau)\,d\tau + \sum_{l\in N_{a,t}} G(\tau_l), \quad f(t) \equiv \int\limits_a^t q(\tau)\,d\tau + \sum_{l\in N_{a,t}} u(\tau_l).$$

We assume that $\det(I_n + G(l)) \neq 0$ $(l = 1,2,\dots)$. Due to the conditions imposed on P, G, q and u, we have $A \in \mathrm{BV}_{loc}([a,b[,\mathbb{R}^{n\times n})$ and $f \in \mathrm{BV}_{loc}([a,b[,\mathbb{R}^n)$. So system (6) is a particular case of system (1), and the impulsive problem (6), (7) to problem (1), (2) for $t_0 = b$.

Along with system (6) consider the perturbed singular system

$$\frac{dx}{dt} = \widetilde{P}(t)x + \widetilde{q}(t), \quad x(\tau_l+) - x(\tau_l-) = \widetilde{G}(l)x(\tau_l) + \widetilde{g}(l) \ \ (l \in \mathbb{N}). \tag{8}$$

Definition 2. Problem (6), (7) is said to be H-well-posed with respect to the matrix-function G_0 if it has a unique solution x and for every $\varepsilon > 0$ there exists $\eta > 0$ such that problem (8), (7) has a unique solution y and estimate (5) holds for every matrix-functions $\widetilde{P}$, $\widetilde{G}$ and vector-functions $\widetilde{q}$, $\widetilde{g}$ such that

$$\begin{gathered}\det(I_n + \widetilde{G}(l)) \neq 0 \ \ (l \in \mathbb{N}),\\ \bigg\|\int\limits_t^{b-} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}_\iota(G_0;\widetilde{P}-P,\widetilde{G}-G))(s)\cdot H(s)\bigg\| < \eta \ \text{ and}\\ \bigg\|\int\limits_t^{b-} H^{-1}(s)\,d\,\mathrm{V}(\mathcal{A}_\iota(G_0;\widetilde{q}-q,\widetilde{g}-g))(s)\bigg\| < \eta \ \text{ for } \ t \in [a,b[.\end{gathered}$$

Theorem 3. *Let there exist a constant matrix $B = (b_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ such that the conditions*

$$\begin{gathered}r(B) < 1, \quad [g_{ii}(l)]_+ > -1 \ \ (i = 1,\dots,n; \ \ l \in \mathbb{N}),\\ c_i(t,\tau) \le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \ \textit{for} \ \ b-\delta \le t \le \tau < b \ \ (i = 1,\dots,n),\end{gathered}$$

$$\bigg|\int\limits_{t}^{b-} c_i(t,\tau)h_i(\tau)[p_{ii}(\tau)]_-\,d\tau+\sum_{l\in N_{t,b}} c_i(t_l,\tau_l)h_i(\tau_l)[g_{ii}(l)]_-\bigg|\le b_{ii}\,h_i(t)\ \ \text{and}$$

$$\bigg|\int\limits_{t}^{b-} c_i(t,\tau)h_k(\tau)\,d\,\mathrm{V}\left(\mathcal{A}_\iota([g_{ii}]_+;p_{ik},g_{ik})\right)(\tau)\bigg|\le b_{ik}\,h_i(t)\ \ \text{for}\ \ t\in b-\delta,b[$$

$(i\neq k;\ i,k=1,\dots,n)$ hold for some $b_0>0$ and $\delta>0$. Let, moreover,

$$\lim_{t\to b-}\int\limits_{t}^{b-}\frac{c_i(t,\tau)}{h_i(t)}\,d\,\mathrm{V}\left(\mathcal{A}_\iota([g_{ii}]_+;q_i,g_i)\right)(\tau)=0\ \ (i=1,\dots,n),$$

where c_i is the Cauchy function of the impulsive equation

$$\frac{dx}{dt}=[p_{ii}(t)]_+\,x,\quad x(\tau_l+)-x(\tau_l-)=[g_{ii}(l)]_+\,x(\tau_l)\ \ (l\in\mathbb{N}).$$

Then problem $(6),(7)$ is H-well-posed with respect to G_0, where

$$G_0(l)\equiv\operatorname{diag}\left([g_{11}(l)]_+,\dots,[g_{nn}(l)]_+\right).$$

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