Coupled-Jumping Timescale Theory and Applications

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Abstract

Recently, the concept of a coupled-jumping timescale space (short for CJTS) $\mathbb{T}_1 - \mathbb{T}_2$ was initiated. Based on it, the theory of calculus and fundamental functions were established. By using this theory, an initial value problem of time-hybrid dynamic equations whose initial value is given in $\mathbb{T}_2$ and the unique solution is located in $\mathbb{T}_1$ can be considered. It is worth noting that the Hilger's theory can be derived through removing the coupled-jumping state by letting $\mathbb{T}_1 = \mathbb{T}_2$ and the Hilger theory is essentially based on a single timescale space. The coupled-jumping timescale theory largely deepens and includes the Hilger theory and brings a completely new significance of dynamic equations on time scales.

1 Vertical evolution of time scales

The time scale theory was introduced by Hilger in 1988 to unify the continuous and discrete analysis (see [1, 2]). This theory plays a very significant role in both pure and applied mathematics, for example, time-hybrid dynamic equations (see [11]), quaternion dynamic equations (see [3,4]), fuzzy dynamic equations (see [5]), the closedness of time scales and related function theory (see [6–9]), stochastic dynamic equations (see [10]) and hybrid measurability theory (see [12]), etc. To further reveal the changing essence of time scales, we first introduced two basic types of the evolution of time scales under which some corresponding dynamic equation were presented (see [11]).

In Figure 1, let $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4\}$ be a timescale group. By Hilger theory, this time scale group will induce a continuous dynamic equation, a piecewise continuous dynamic equation, a discrete dynamic equation and a quantum dynamic equation in sequence. Starting with the evolution process of these time scales, $\mathbb{T}$ varies from the form $\mathbb{T}_1$ to the form $\mathbb{T}_4$ in the timescale group, such a **vertical evolution** in the timescale group acts as a direct factor which leads to the four different types of dynamic equation during the changing process of the time scale $\mathbb{T}$. Only when $\mathbb{T}$ is fixed in this timescale group, the concrete dynamic equation can be determined. From the viewpoint of the evolution process of time scales, the essence of Hilger's theory depends on the vertical evolution of time scales, accordingly the unification of various types of dynamic equation can be achieved when the form of $\mathbb{T}$ is fixed in a timescale group. In other words, the related analysis and applications on Hilger theory are purely based on a **single time scale** during this evolution.

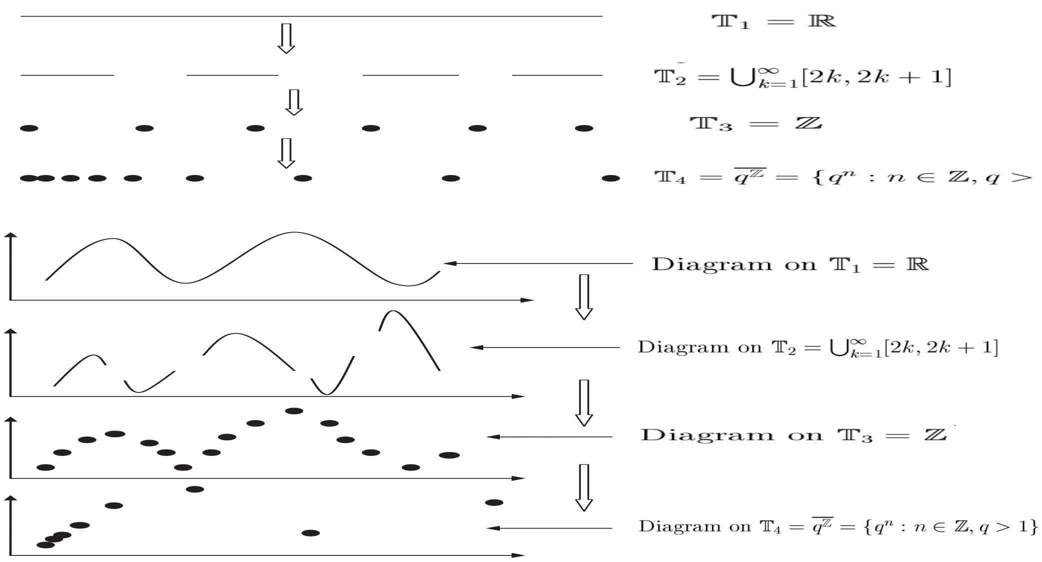

Figure 1. The vertical evolution diagram of dynamical behavior from $\mathbb{T}_1$ to $\mathbb{T}_4$ under Hilger theory

2 Hybrid-timescale problems-a horizontal evolution of time scales

The other natural and significant evolution of time scales that must be referred to is **horizontal evolution** of time scales. The related problems caused by horizontal evolution of time scales cannot be solved by Hilger theory and they still belong to the problems of timescale category. In Figure 2, let

$$\mathbb{T}_1 = \overline{\{q^n : q > 1, n \in \mathbb{Z}^- \cup \{0\}\}}, \quad \mathbb{T}_2 = [1.1, 3.7], \quad \mathbb{T}_3 = \bigcup_{k=2}^{5}[2k, 2k+1],$$
$$\mathbb{T}_4 = \{12.1, 13.1, 14.1, 15.1, 16.1\}, \quad \mathbb{T}_5 = \overline{\{(1.5)^n : n \geq 7\}}, \dots .$$

For convenience, let a timescale group be formed by $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5, \dots\}$. It is easy to observe that the dynamical behavior described by Figure 2 exists on the time scale $\mathbb{T}$ formed by five districts and each district is a time scale, i.e., $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \cup \mathbb{T}_4 \cup \mathbb{T}_5 \cup \dots$. Therefore, the switch of the dynamical behavior in four timescale districts is directly caused by a **horizontal evolution** of all the time scales in this timescale group.

Usually, all the similar problems described by Figures 2 are called the **hybrid-timescale problems**. Essentially, the hybrid-timescale problems are formed by the problems on multiple time scales and this class of problems can be precisely depicted by a **horizontal evolution** of time scales in a timescale group.

By comparison, the related hybrid-timescale problems are more comprehensive and will strictly include the problems on a single time scale as their particular cases (see Figure 3 for their detailed

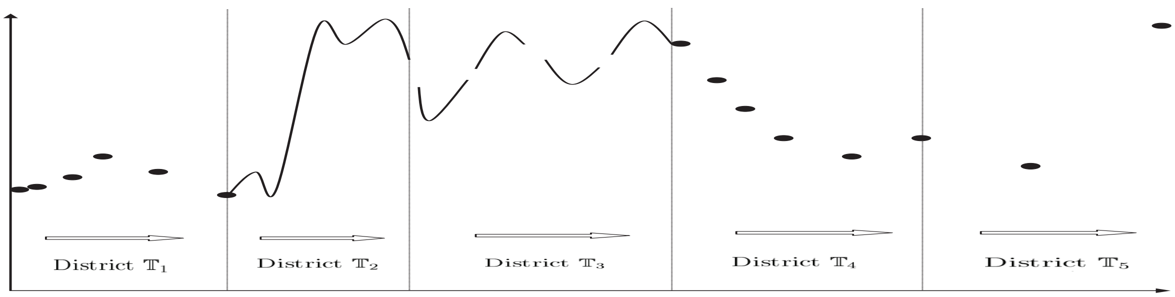

Figure 2. The horizontal evolution diagram of dynamical behavior from $\mathbb{T}_1$ to $\mathbb{T}_4$ under coupled-jumping timescale theory

relations). Moreover, the dynamical behavior on hybrid time scales cannot be effectively studied purely on a single time scale through Hilger theory. Therefore, it is very necessary to establish a theory (we call it coupled-jumping timescale theory) to solve the hybrid-timescale problems.

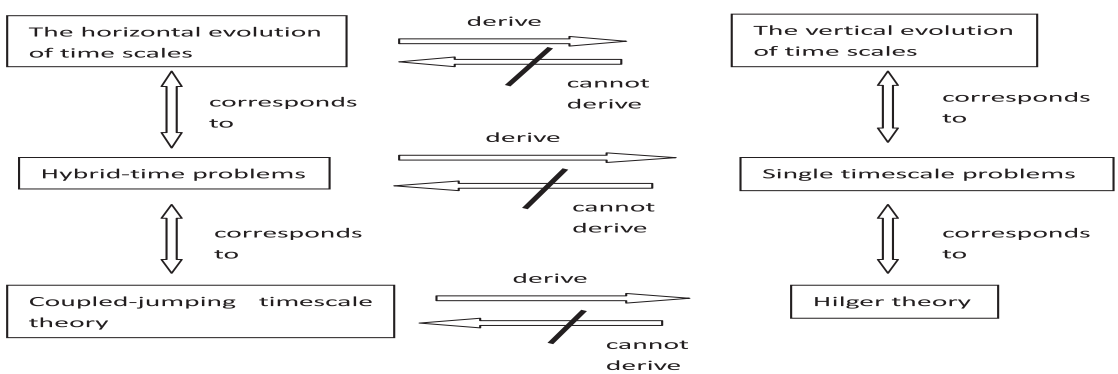

Figure 3. The relation among hybrid-timescale problems, single-timescale problems, Hilger theory and Coupled-jumping timescale theory

3 The description of the hybrid-timescale initial-value problems

For understanding the idea to solve the hybrid-timescale problems, we will adopt Figure 2 to illustrate our methods and the framework of the solving steps. Let a timescale group be $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5, \dots\}$. To break through the limitation of the Hilger theory and to establish a coupled-jumping timescale theory, demonstrating a distinct dynamical behavior on time scales, firstly, we must consider the formation process of the dynamical behavior in Figure 2. Assume that the dynamical behavior in Figure 2 corresponds to a solution $x(t)$ of a dynamic equation on the hybrid time scales with the initial point $(t_0, x(t_0))$, where $t_0 = 0 \in \mathbb{T}_1$. According to the continuous dependence on initial values of solutions and the continuation theorem, there is a solution on the district $\mathbb{T}_1$ such that $(t_1, x(t_1))$ is the right boundary point on the district $\mathbb{T}_1$, where $t_1 = 1 \notin \mathbb{T}_2$. Now taking $(t_1, x(t_1))$ as the initial point, there is a solution on the district $\mathbb{T}_2$ such that $(t_2, x(t_2))$ is the right boundary point on the district $\mathbb{T}_2$, where $t_2 = 3.7 \notin \mathbb{T}_3$. Next, by taking $(t_2, x(t_2))$ as the initial point, there is a solution on the district $\mathbb{T}_3$ such that $(t_3, x(t_3))$ is the right boundary point on the district $\mathbb{T}_3$, where $t_3 = 11 \notin \mathbb{T}_4$. Repeating the process, by taking $(t_3, x(t_3))$ as the initial point, there is a solution on the district $\mathbb{T}_4$ such that $(t_4, x(t_4))$ is the right boundary point on the district $\mathbb{T}_4$, where $t_4 = 16.1 \notin \mathbb{T}_5$. Finally, the solution on the district $\mathbb{T}_5$ is determined by the initial point $(t_4, x(t_4))$. If there are more time scales after $\mathbb{T}_5$, for instance, $\mathbb{T}_6, \mathbb{T}_7, \dots$, the process above can be continued until the solution exists on $\mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \cdots := \bigcup_{i=1}^{+\infty} \mathbb{T}_i$.

In the above process, a key problem appears. Note that $t_1 \notin \mathbb{T}_2$ but the solution on district $\mathbb{T}_2$ is continuously dependent on $(t_1, x(t_1))$, similarly, $t_2 \notin \mathbb{T}_3$ but the solution on district $\mathbb{T}_3$ is continuously dependent on $(t_2, x(t_2))$,..., $t_4 \notin \mathbb{T}_5$ but the solution on district $\mathbb{T}_5$ is continuously dependent on $(t_4, x(t_4)), \dots$. Therefore, the first problem we must solve is that we should introduce an initial value problem of a dynamic equations whose initial value is given in one time scale and the unique solution is located in another. In the literature [11], the coupled-jumping timescale theory (or hybrid-timescale theory) was proposed.

4 The coupled-jumping timescale space (CJTS) and calculus

We present a notion of coupled-jumping timescale space and a concept of the hybrid-composition integral.

Definition 4.1 ([2]). For $\widehat{t} \in \mathbb{T}_k$, we define the forward jump operator $\sigma_k : \mathbb{T}_k \to \mathbb{T}_k$ by $\sigma_k(\widehat{t}) = \inf\{s \in \mathbb{T}_k : \ s > \widehat{t}\}$; the backward jump operator $\rho_k : \mathbb{T}_k \to \mathbb{T}_k$ by $\rho_k(\widehat{t}) = \sup\{s \in \mathbb{T}_k : \ s < \widehat{t}\}$; and the graininess function $\mu_k : \mathbb{T}_k \to [0, +\infty)$ by $\mu_k(\widehat{t}) = \sigma_k(\widehat{t}) - \widehat{t}$, where $k = 1, 2$.

Now, we will introduce the jumping construction of the coupled-jumping timescale space $\mathbb{T}_1 - \mathbb{T}_2$.

Definition 4.2 ([11]). Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be a pair of time scales. For $t \in \mathbb{T}_1 \cup \mathbb{T}_2$, we define the coupled-forward jump operator between $\mathbb{T}_1$ and $\mathbb{T}_2$ by $\sigma_{\mathbb{T}_2}(t) = \inf\{s \in \mathbb{T}_2 : \ s \geq t\}$, and define the coupled-backward jump operator between $\mathbb{T}_1$ and $\mathbb{T}_2$ by $\rho_{\mathbb{T}_2}(t) = \sup\{s \in \mathbb{T}_2 : \ s \leq t\}$. We say t is a coupled right-dense point iff $\sigma_{\mathbb{T}_2}(t) = t$; t is a coupled right-scattered point iff $\sigma_{\mathbb{T}_2}(t) > t$; t is a coupled left-dense point iff $\rho_{\mathbb{T}_2}(t) = t$; t is a coupled left-scattered point iff $\rho_{\mathbb{T}_2}(t) < t$; t is a coupled isolated point iff $\rho_{\mathbb{T}_2}(t) < t < \sigma_{\mathbb{T}_2}(t)$ (see Figure 4).

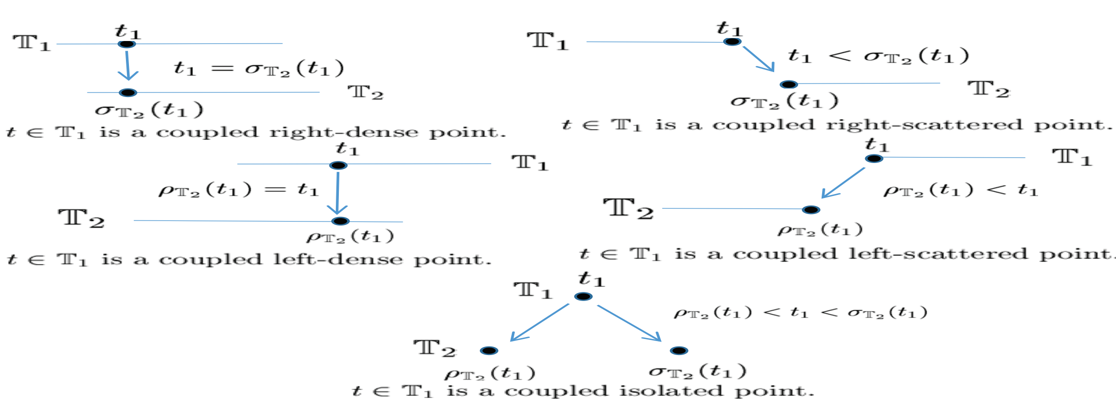

Figure 4. Schematic diagram of all types of coupled-jumping points

Definition 4.3 ([11]). Let $f : \mathbb{T}_1 \cup \mathbb{T}_2 \to \mathbb{R}$. We define a hybrid-composition integral (or short for

HC-integral) of $f(t)$ on CJTS as follows:

$$\int\limits_a^b f(\tau)\Delta_m\tau = \begin{cases} \alpha \int [\sigma_{\mathbb{T}_1}(a), \rho_{\mathbb{T}_1}(b)]_{\mathbb{T}_1} f(\tau)\Delta_1\tau + (1-\alpha) \displaystyle\int\limits_{[\sigma_{\mathbb{T}_2}(a),\rho_{\mathbb{T}_2}(b)]_{\mathbb{T}_2}} f(\tau)\Delta_2\tau, & a < b, \\ -\alpha \displaystyle\int\limits_{[\sigma_{\mathbb{T}_1}(b),\rho_{\mathbb{T}_1}(a)]_{\mathbb{T}_1}} f(\tau)\Delta_1\tau - (1-\alpha) \int [\sigma_{\mathbb{T}_2}(b), \rho_{\mathbb{T}_2}(a)]_{\mathbb{T}_2} f(\tau)\Delta_2\tau, & a > b, \end{cases}$$

where $a, b \in \mathbb{T}_1 \cup \mathbb{T}_2$, $0 \le \alpha \le 1$ and α is called the hybrid-composition proportion coefficient.

5 Time-hybrid dynamic equations on CJTS

In this section, we will introduce the exponential function on coupled-jumping time scales and introduce the basic theorem of time-hybrid dynamic equations. For more details, one may consult the literature [11].

Definition 5.1 ([11])**.** Let $\ddot{t}, s \in \mathbb{T}_1 \cup \mathbb{T}_2$. We introduce the HC-exponential function by

$$\overline{e}_f(\ddot{t}, s) := \begin{cases} \exp\bigg\{\alpha \displaystyle\int\limits_{[\sigma_{\mathbb{T}_1}(s),\rho_{\mathbb{T}_1}(\ddot{t})]_{\mathbb{T}_1}} \frac{\operatorname{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\,\Delta_1\tau \\ \qquad\qquad +(1-\alpha) \displaystyle\int\limits_{[\sigma_{\mathbb{T}_2}(s),\rho_{\mathbb{T}_2}(\ddot{t})]_{\mathbb{T}_2}} \frac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\,\Delta_2\tau\bigg\}, & s < \ddot{t}, \\ \exp\bigg\{-\alpha \displaystyle\int\limits_{[\sigma_{\mathbb{T}_1}(\ddot{t}),\rho_{\mathbb{T}_1}(s)]_{\mathbb{T}_1}} \frac{\operatorname{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\,\Delta_1\tau \\ \qquad\qquad -(1-\alpha) \displaystyle\int\limits_{[\sigma_{\mathbb{T}_2}(\ddot{t}),\rho_{\mathbb{T}_2}(s)]_{\mathbb{T}_2}} \frac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\,\Delta_2\tau\bigg\}, & s > \ddot{t}. \end{cases}$$

In the following theorem, we will demonstrate the HC-exponential solution of the homogeneous time-hybrid dynamic equation.

Theorem 5.1 ([11])**.** *Let $t \in \mathbb{T}_1^{\overline{\kappa}}$, $s \in \mathbb{T}_2^{\overline{\kappa}}$, $t \ge s$. Then $\overline{e}_f(t,s)$ is the solution of the initial value problem*

$$\mu_1(t)x^{\Delta_t}(t) = \bigg\{\big(1+\mu_1(t)f(t)\big)^\alpha \exp\bigg\{(1-\alpha)\int\limits_{\rho_{\mathbb{T}_2}(t)}^{\rho_{\mathbb{T}_2}(\sigma_1(t))} \frac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\Delta_2\tau\bigg\} - 1\bigg\}x(t), \quad (5.1)$$

with the initial value $x(s) = 1$, where $x^{\Delta_t}(t)$ denotes the Δ-derivative at t on $\mathbb{T}_1$.

Remark 5.1. Notice that the initial value problem of the homogeneous time-hybrid dynamic equation (5.1) has the characteristic that the initial value is given in $\mathbb{T}_2$ and the unique solution is located in $\mathbb{T}_1$, where $\mathbb{T}_1$ **may not be equal to** $\mathbb{T}_2$. There has been no theory to support the study of such a type of time-hybrid dynamic equation before now.

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