Topological Entropy of a Diagonalizable Linear System of Differential Equations

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Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, \infty), \tag{1}$$

with a piecewise continuous operator function $A : \mathbb{R}_+ \to \operatorname{End} \mathbb{R}^n$. Let us endow the space $\mathbb{R}^n$ with the norm $\|x\| = \max\limits_{1 \leqslant k \leqslant n} |x_k|$ and with the set of metrics

$$d_t^A(x_0, y_0) = \max_{\tau \in [0,t]} \|x(\tau, x_0) - x(\tau, y_0)\|, \quad x_0, y_0 \in \mathbb{R}^n, \quad t \in \mathbb{R}_+,$$

where $x(\,\cdot\,, a)$ is the solution to system (1) satisfying the condition $x(0, a) = a$. By $S_{\|\cdot\|}(A, \mathcal{K}, \varepsilon, t)$ we denote the *ε-entropy* of a compact metric space $\mathcal{K} \subset \mathbb{R}^n$ with the metric d_t^A [2] (that is, the minimum number of open balls of radius $\varepsilon > 0$ covering $\mathcal{K}$). Then the *topological entropy* [1] of system (1) is defined by the formula

$$h_{\rm top}(A) = \sup_{\mathcal{K} \subset \mathbb{R}^n} \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \frac{1}{t} \ln S_{\|\cdot\|}(A, \mathcal{K}, \varepsilon, t)$$

(its right-hand side does not depend on the choice of a norm $\|\cdot\|$, therefore the definition is correct).

In what follows we will use one more formula to calculate the topological entropy. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ we denote by $N_{\|\cdot\|}(A, \mathcal{K}, \varepsilon, t)$ the *ε-capacity* of a compact metric space $\mathcal{K} \subset \mathbb{R}^n$ with the metric d_t^A [2] (i.e., the maximum number of points such that all their pairwise d_t^A-distances are greater than ε), then the topological entropy can be calculated by the formula

$$h_{\rm top}(A) = \sup_{\mathcal{K} \subset \mathbb{R}^n} \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \frac{1}{t} \ln N_{\|\cdot\|}(A, \mathcal{K}, \varepsilon, t).$$

In [3], it is asserted that for the Lyapunov exponents $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ of any system (1) with a bounded operator function A, the equality

$$h_{\rm top}(A) = \sum_{\lambda_i(A) > 0} \lambda_i(A) \tag{2}$$

holds. In fact, it may not hold, as shown by

Theorem 1. *For system* (1) *with the operator function*

$$A(t) = \operatorname{diag}(a(t), b(t)), \quad \text{where} \;\; (a(t), b(t)) = \begin{cases} (1,0), & t \in [0,1]; \\ (1,0), & t \in [(2n-1)!, (2n)!]; \\ (0,1), & t \in [(2n)!, (2n+1)!], \end{cases} \quad n = 1, 2, \dots, \tag{3}$$

relation (2) *becomes the inequality* $1 < 2$.

Proof. Let us calculate the Lyapunov exponents of system (3). On the one hand, from the inequalities $a(t) \leqslant 1$, $b(t) \leqslant 1$, $t \geq 0$, we see that the Lyapunov exponents do not exceed 1.

On the other hand, from the inequalities

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_0^t a(\tau)\,d\tau \geqslant \lim_{n\to+\infty} \frac{(2n)! - (2n-1)!}{(2n)!} = 1,$$

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_0^t b(\tau)\,d\tau \geqslant \lim_{n\to+\infty} \frac{(2n+1)! - (2n)!}{(2n+1)!} = 1,$$

it follows that the Lyapunov exponents of system (3) satisfy the equalities

$$\lambda_1(A) = \lambda_2(A) = 1 \ \text{ and } \ \lambda_1(A) + \lambda_2(A) = 2.$$

Let us calculate the topological entropy of system (3). For given $m, n \in \mathbb{N}$, consider the set of points of the form $\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}$, where

$$x_1^{(k)} = \frac{k}{e^{(2n)!-(2n-1)!}m}\,, \quad k = 0, \dots, [e^{(2n)!-(2n-1)!}m]$$

($[\,\cdot\,]$ is an integer part of the number). Since the distance $d^A_{(2n)!}\Big(\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^{(l)} \\ 0 \end{pmatrix}\Big)$, $k \neq l$, satisfies the inequality

$$d^A_{(2n)!}\bigg(\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^{(l)} \\ 0 \end{pmatrix}\bigg) = \frac{|k-l|}{e^{(2n)!-(2n-1)!}m}\, e^{(2n)!-(2n-1)!+(2n-2)!-(2n-3)!+\cdots} > \frac{1}{m}\,,$$

then

$$N\Big(A, [0,1] \times \{0\}, \frac{1}{m}\,, (2n)!\Big) \geqslant e^{(2n)!-(2n-1)!}m,$$

and hence

$$h_{\mathrm{top}}(A) \geqslant \lim_{n\to\infty} \Big(\frac{(2n)! - (2n-1)!}{(2n)!} + \frac{\ln m}{(2n)!}\Big) = 1.$$

Let us prove the opposite inequality $h_{\mathrm{top}}(A) \leqslant 1$. For an arbitrary compact set $K \subset \mathbb{R}^2$ we denote by γ_K a positive number such that $K \subset [-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$.

For any $t > 0$ and $m \in \mathbb{N}$, the set of points $V_{t,m}$ of the form $\begin{pmatrix} x_1^{(k)} \\ x_2^{(l)} \end{pmatrix}$, where

$$x_1^{(k)} = \frac{k\gamma_K}{m}\, e^{-\int_0^t a(\tau)\,d\tau}, \quad k = -\Big[e^{\int_0^t a(\tau)\,d\tau} m\Big], \dots, \Big[e^{\int_0^t a(\tau)\,d\tau} m\Big],$$

$$x_2^{(l)} = \frac{l\gamma_K}{m} e^{-\int_0^t b(\tau)\,d\tau}, \quad l = -\Big[e^{\int_0^t b(\tau)\,d\tau} m\Big], \dots, \Big[e^{\int_0^t b(\tau)\,d\tau} m\Big],$$

is a $\frac{\gamma_K}{m}$-covering of the square $[-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$. Indeed, let an arbitrary point $(x_1, x_2) \in [-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$ be given. Then by the definition of the set $V_{t,m}$ there exists a point $(x_1^{(k_0)}, x_2^{(l_0)})$ such that

$$|x_1^{(k_0)} - x_1| \leqslant \frac{\gamma_K}{m}\, e^{-\int_0^t a(\tau)\,d\tau}, \quad |x_2^{(l_0)} - x_2| \leqslant \frac{\gamma_K}{m}\, e^{-\int_0^t b(\tau)\,d\tau}.$$

It follows that
$$|x_1^{(k_0)} - x_1|e^{\int\limits_0^t a(\tau)\,d\tau} \leqslant \frac{\gamma_K}{m}\,, \quad |x_2^{(l_0)} - x_2|e^{\int\limits_0^t b(\tau)\,d\tau} \leqslant \frac{\gamma_K}{m}\,.$$
Thus $S_{\|\,\cdot\,\|}(A,\mathcal{K},\varepsilon,t)$ does not exceed the cardinality of the set $V_{t,m}$, which equals
$$\Big(2\big[e^{\int\limits_0^t a(\tau)\,d\tau}m\big]+1\Big)\Big(2\big[e^{\int\limits_0^t b(\tau)\,d\tau}m\big]+1\Big) \leqslant 9m^2e^{\int\limits_0^t (a(\tau)+b(\tau))\,d\tau} = 9m^2e^t,$$
whence we get $h_{\rm top}(A) \leqslant 1$. □

For an arbitrary piecewise continuous function $a(\,\cdot\,):\mathbb{R}_+\to\mathbb{R}$, let
$$a^+(t) = \max_{s\in[0;t]}\int\limits_0^s a(\tau)\,d\tau.$$

Theorem 2. *If system* (1) *can be reduced to a diagonal form*
$$\dot y = \operatorname{diag}(b_1(t),\dots,b_n(t))y,$$
by means of a transformation $x = Q(t)y$ such that
$$\varlimsup_{t\to+\infty}\frac{1}{t}\,\ln\|Q(t)\| = \varlimsup_{t\to+\infty}\frac{1}{t}\,\ln\|Q^{-1}(t)\| = 0,$$
then
$$h_{\rm top}(A) = \varlimsup_{t\to+\infty}\frac{1}{t}\sum_{k=1}^n b^+(t).$$

Given a metric space $\mathcal{M}$ and a continuous map
$$A:\mathcal{M}\times\mathbb{R}_+\to\operatorname{End}\mathbb{R}^n \tag{4}$$
we form the function
$$\mu\longmapsto h_{\rm top}(A(\mu,\,\cdot\,)). \tag{5}$$
Results of [4, 5] imply

Theorem 3. *For any mapping* (4), *function* (5) *belongs to the third Baire class. If $\mathcal{M}=[0,1]$, then for some mapping* (4) *function* (5) *does not belong to the first Baire class.*

References

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